

MATEMATIČKA ANALIZA

Površina lika
omeđenog
krivuljama

Newton – Leibnizova formula

Neka je funkcija f definirana i neprekidna na $[a, b]$.
Tada postoji primitivna funkcija F funkcije f i vrijedi:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Metode integriranja

- Neposredna integracija
- Metoda supstitucije
 - zamjena granica integrala
- Parcijalna integracija

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Zadatak 1. $\int_0^2 x^3 \sqrt{x^4 + 9} dx$

$$= \left| \begin{array}{l} x^4 + 9 = t \\ 4x^3 dx = dt \\ dx = \frac{dt}{4x^3} \\ x = 0 \rightarrow t = 0^4 + 9 = 9 \\ x = 2 \rightarrow t = 2^4 + 9 = 25 \end{array} \right| = \int_9^{25} x^3 \sqrt{t} \frac{dt}{4x^3}$$

$$= \frac{1}{4} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \Big|_9^{25} = \frac{1}{6} \sqrt{t^3} \Big|_9^{25} = \frac{125}{6} - \frac{27}{6} = \frac{49}{3}$$

Zadatak 2. $\int_0^2 x^3 \sqrt{x^4 + 9} dx = (*)$

$$\int x^3 \sqrt{x^4 + 9} dx = \left| \begin{array}{l} x^4 + 9 = t \\ 4x^3 dx = dt \\ dx = \frac{dt}{4x^3} \end{array} \right| = \int x^3 \sqrt{t} \frac{dt}{4x^3}$$

$$= \frac{1}{4} \int \sqrt{t} dt = \frac{1}{4} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{6} \sqrt{t^3} + c = \frac{1}{6} \sqrt{(x^4 + 9)^3} + c$$

$$(*) = \frac{1}{6} \sqrt{(x^4 + 9)^3} \Big|_0^2 = \frac{1}{6} \sqrt{25^3} - \frac{1}{6} \sqrt{9^3} = \frac{49}{3}$$

Zadatak 3.

$$\int_1^e \frac{\ln x}{x^3} dx = \left| \begin{array}{l} u = \ln x \quad dv = x^{-3} dx \\ du = \frac{1}{x} dx \quad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} \end{array} \right|$$

$$= \left(-\frac{1}{2x^2} \ln x \right) \Big|_1^e - \int_1^e -\frac{1}{2x^2} \cdot \frac{1}{x} dx$$

$$= \left(-\frac{1}{2x^2} \ln x \right) \Big|_1^e + \frac{1}{2} \int_1^e x^{-3} dx$$

Zadatak 3.

$$= \left(-\frac{1}{2x^2} \ln x \right) \Big|_1^e + \frac{1}{2} \int_1^e x^{-3} dx$$

$$= \left(-\frac{\ln x}{2x^2} \right) \Big|_1^e + \frac{1}{2} \cdot \frac{x^{-2}}{-2} \Big|_1^e = \left(-\frac{\ln x}{2x^2} \right) \Big|_1^e - \frac{1}{4x^2} \Big|_1^e$$

$$= \left(-\frac{\ln e}{2e^2} \right) - \left(-\frac{\ln 1}{2 \cdot 1^2} \right) - \left(\frac{1}{4e^2} - \frac{1}{4 \cdot 1^2} \right)$$

$$= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} = -\frac{3}{4e^2} + \frac{1}{4}$$

Zadatak 4. $\int_{\frac{\pi}{2}}^{\pi} (x - 3) \cos 2x \, dx$

$$\int (x - 3) \cos 2x \, dx = \left| \begin{array}{ll} u = x - 3 & dv = \cos 2x \, dx \\ du = dx & v = \frac{1}{2} \sin 2x \end{array} \right|$$

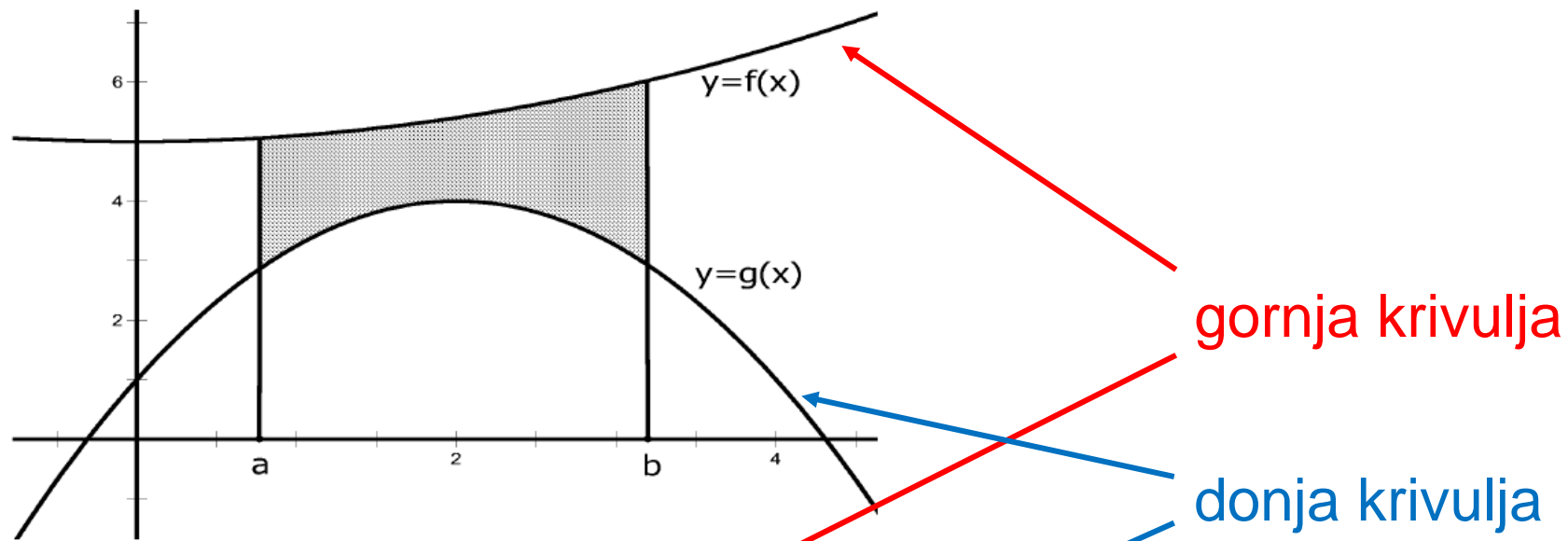
$$= \frac{x - 3}{2} \sin 2x - \frac{1}{2} \int \sin 2x \, dx$$

$$= \frac{x - 3}{2} \sin 2x + \frac{1}{4} \cos 2x + c$$

Zadatak 4.

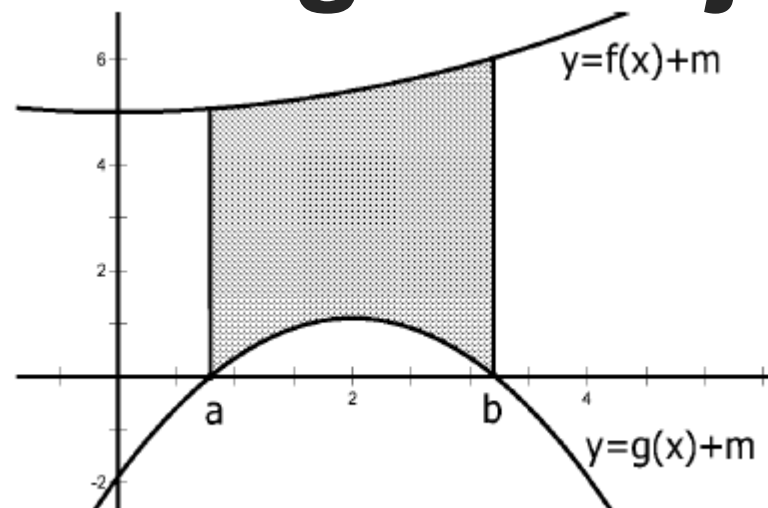
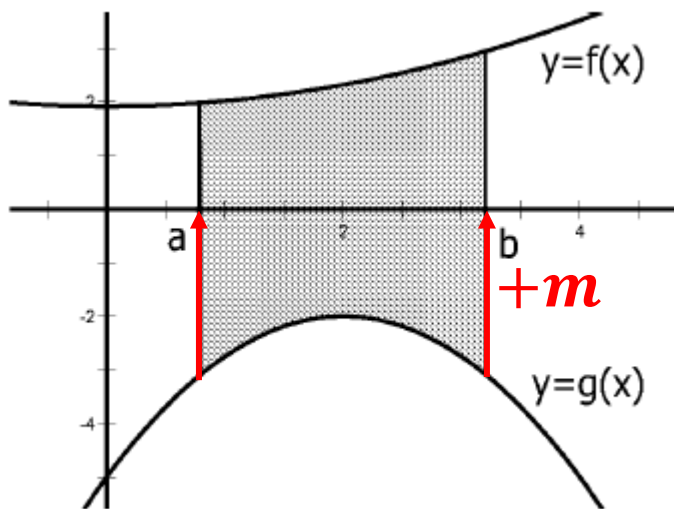
$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} (x - 3) \cos 2x \, dx &= \left(\frac{x - 3}{2} \sin 2x + \frac{1}{4} \cos 2x \right) \Big|_{\frac{\pi}{2}}^{\pi} \\ &= \left(\frac{\pi - 3}{2} \sin 2\pi + \frac{1}{4} \cos 2\pi \right) - \left(\frac{\frac{\pi}{2} - 3}{2} \sin \pi + \frac{1}{4} \cos \pi \right) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\end{aligned}$$

Površina lika omeđenog krivuljama



$$P = \int_a^b (f(x) - g(x)) dx$$

Površina lika omeđenog krivuljama



$$P = \int_a^b ((f(x) + m) - (g(x) + m)) dx$$

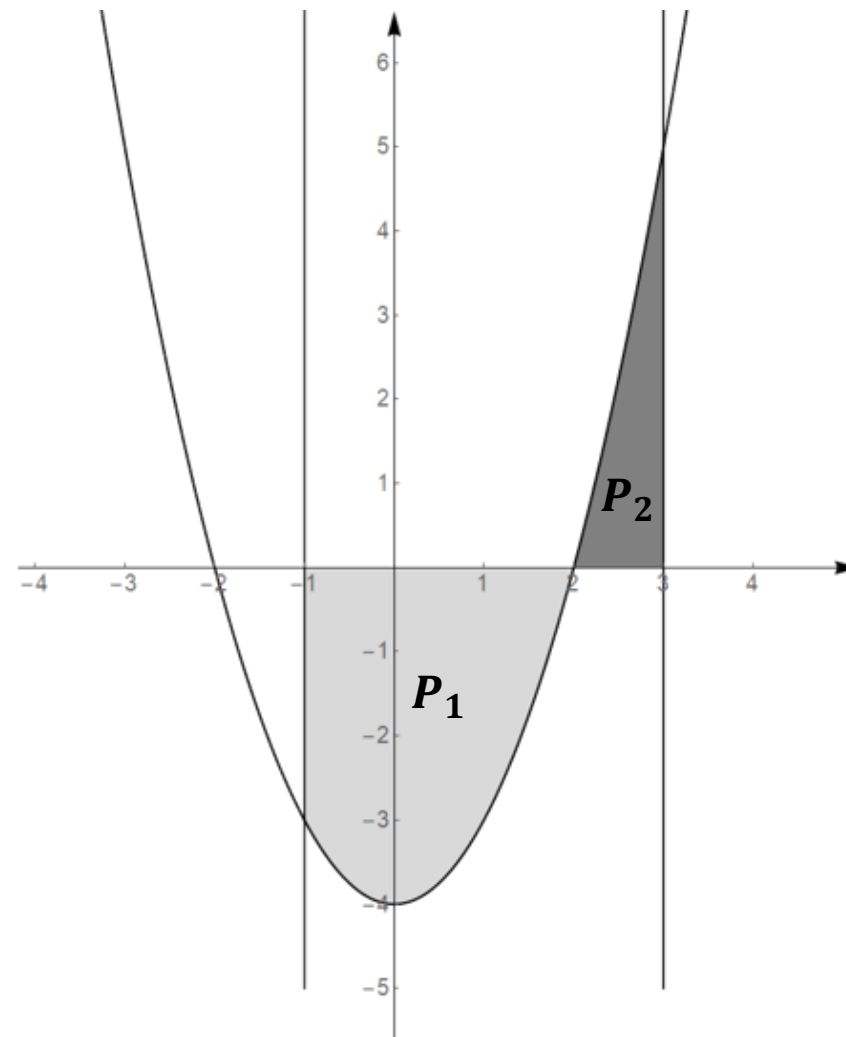
$$P = \int_a^b (f(x) - g(x)) dx$$

Zadatak 5. Izračunajte površinu lika omeđenog krivuljama:
 $y = x^2 - 4$, $x = -1$, $x = 3$, $y = 0$.

$$P_1 = \int_{-1}^2 (0 - (x^2 - 4)) dx$$

$$= \left(-\frac{x^3}{3} + 4x \right) \Big|_{-1}^2$$

$$= 9$$

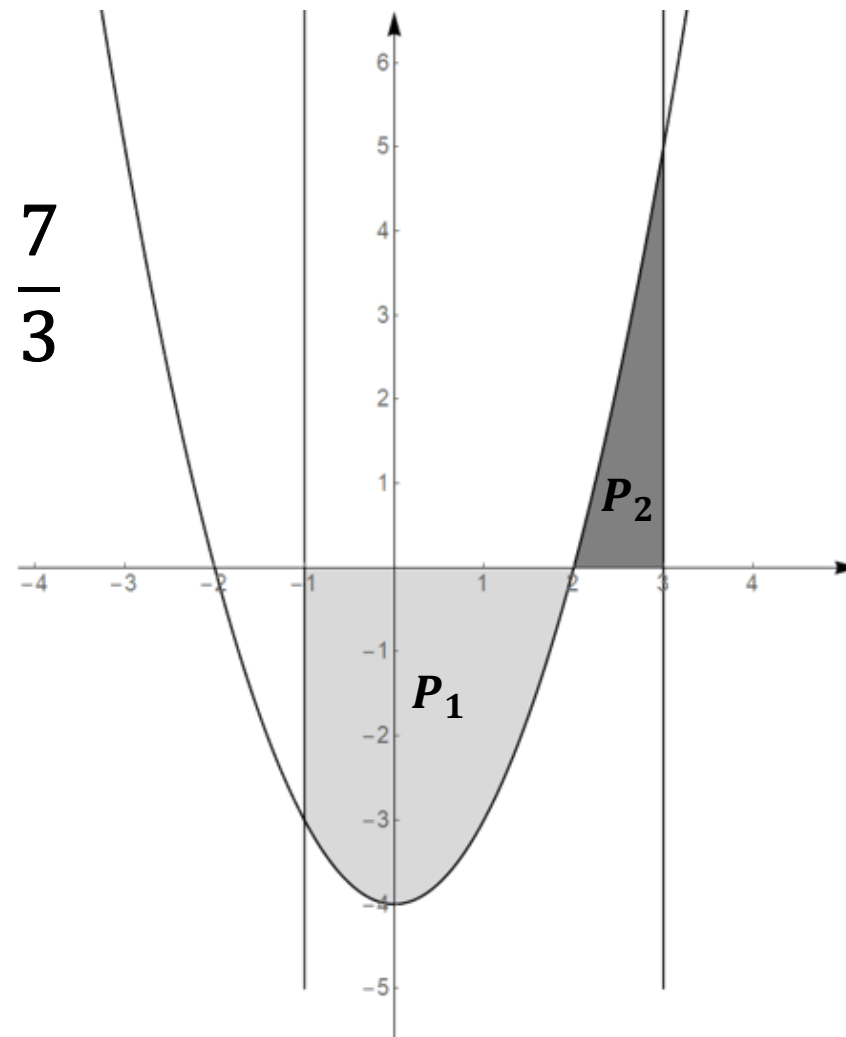


Zadatak 5. Izračunajte površinu lika omeđenog krivuljama:
 $y = x^2 - 4$, $x = -1$, $x = 3$, $y = 0$.

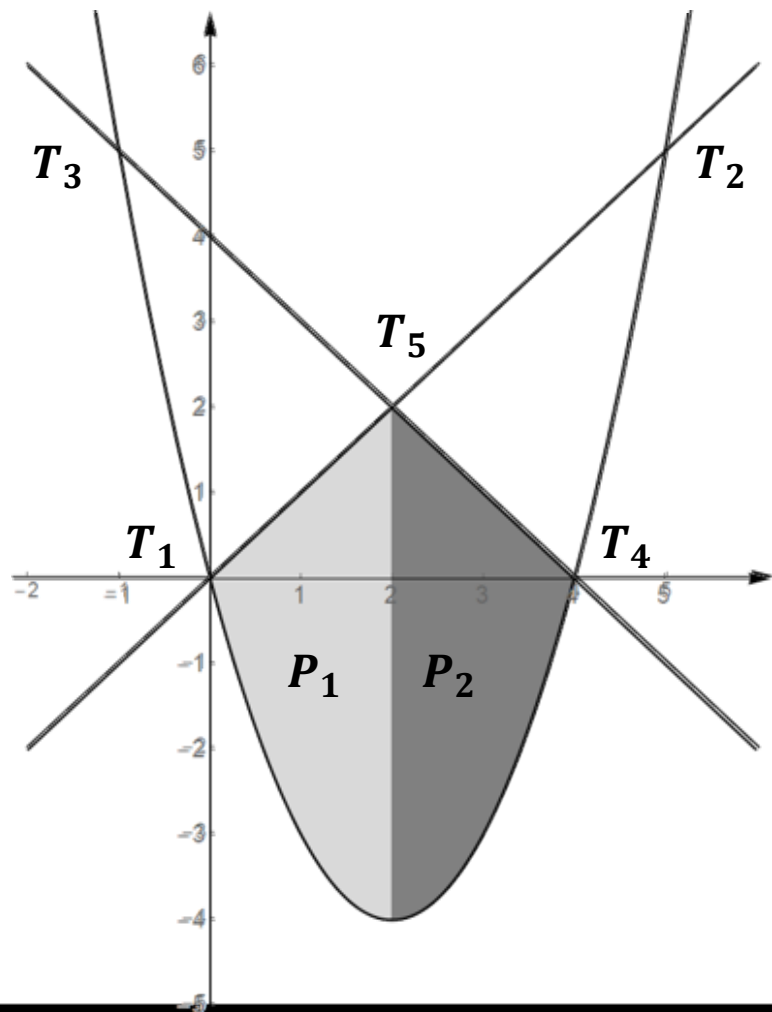
$$P_2 = \int_2^3 (x^2 - 4 - 0) dx = \left(\frac{x^3}{3} - 4x \right) \Big|_2^3 = \frac{7}{3}$$

$$P_1 + P_2 = \frac{34}{3}$$

$$\int_{-1}^3 (x^2 - 4) dx = -\frac{20}{3}$$



Zadatak 5. Izračunajte površinu najvećeg lika omeđenog krivuljama: $y = x$, $y = x^2 - 4x$, $y = 4 - x$.



Presjecišta: $y = x$, $y = x^2 - 4x$

$$x^2 - 4x = x$$

$$x_1 = 0$$

$$x_2 = 5$$

$$y_1 = 0$$

$$y_2 = 5$$

$$T_1(0,0)$$

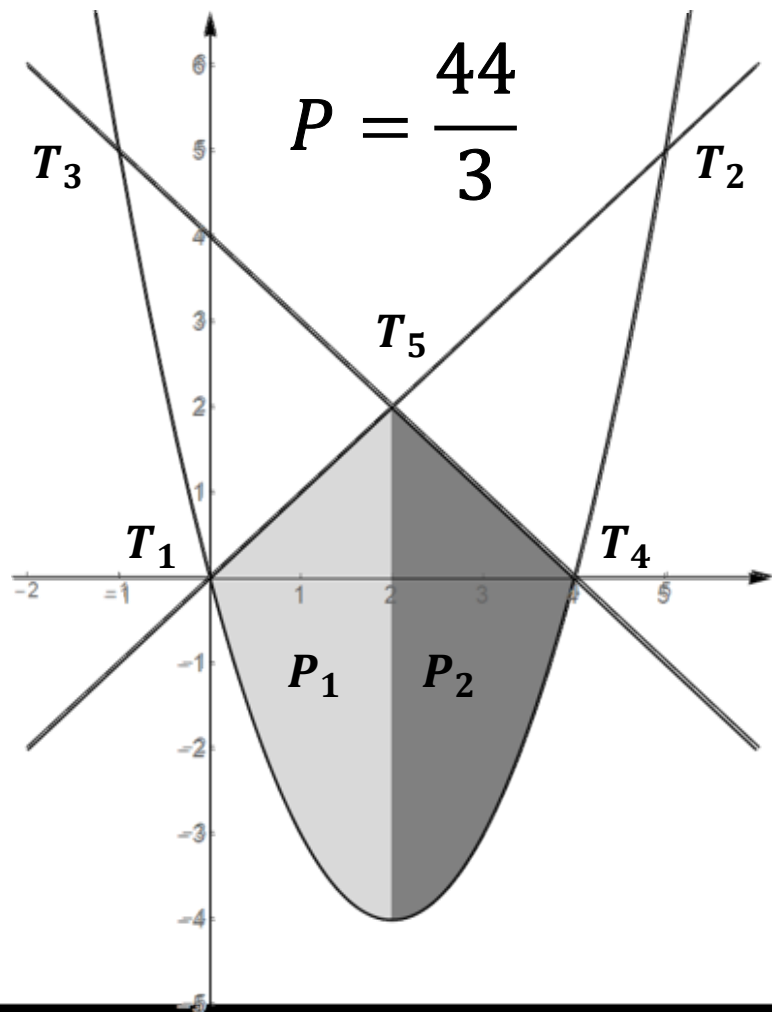
$$T_2(5,5)$$

$$T_3(-1,5)$$

$$T_4(4,0)$$

$$T_5(2,2)$$

Zadatak 5. Izračunajte površinu najvećeg lika omeđenog krivuljama: $y = x$, $y = x^2 - 4x$, $y = 4 - x$.



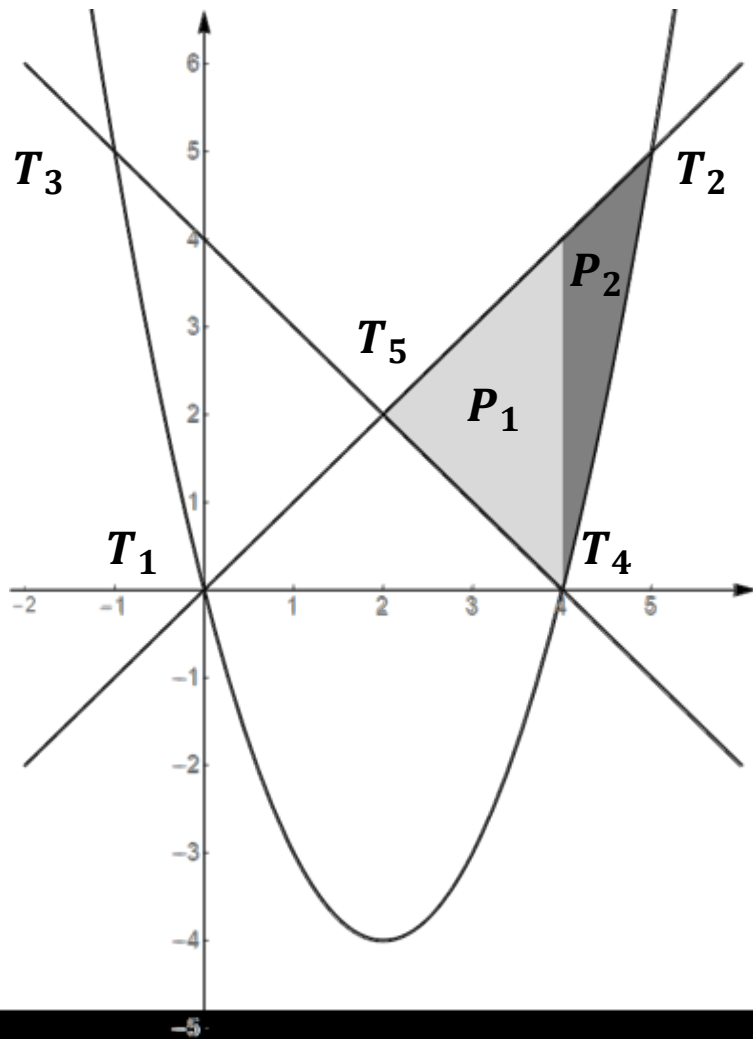
$$P_1 = \int_0^2 (x - (x^2 - 4x)) dx$$

$$= \left(5\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^2 = \frac{22}{3}$$

$$P_2 = \int_2^4 ((4 - x) - (x^2 - 4x)) dx$$

$$= \left(4x + 3\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_2^4 = \frac{22}{3}$$

Zadatak 5. Izračunajte površinu lika u prvom kvadrantu omeđenog krivuljama: $y = x$, $y = x^2 - 4x$, $y = 4 - x$.



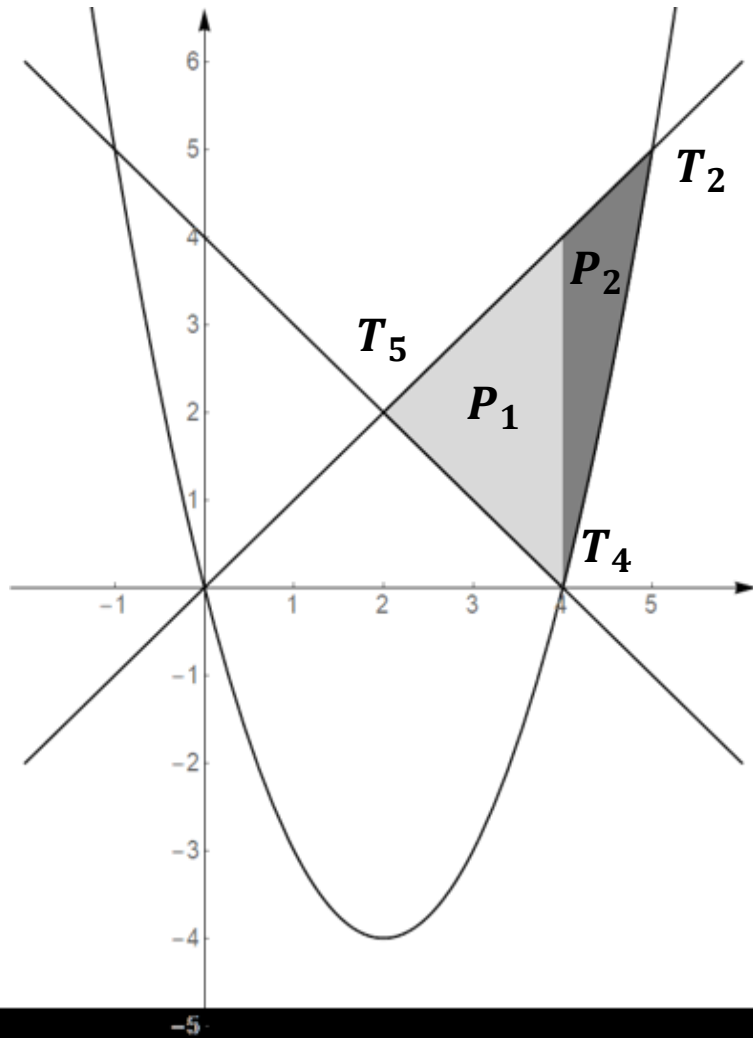
Presjecišta:

$$T_2(5,5)$$

$$T_4(4,0)$$

$$T_5(2,2)$$

Zadatak 5. Izračunajte površinu lika u prvom kvadrantu omeđenog krivuljama: $y = x$, $y = x^2 - 4x$, $y = 4 - x$.

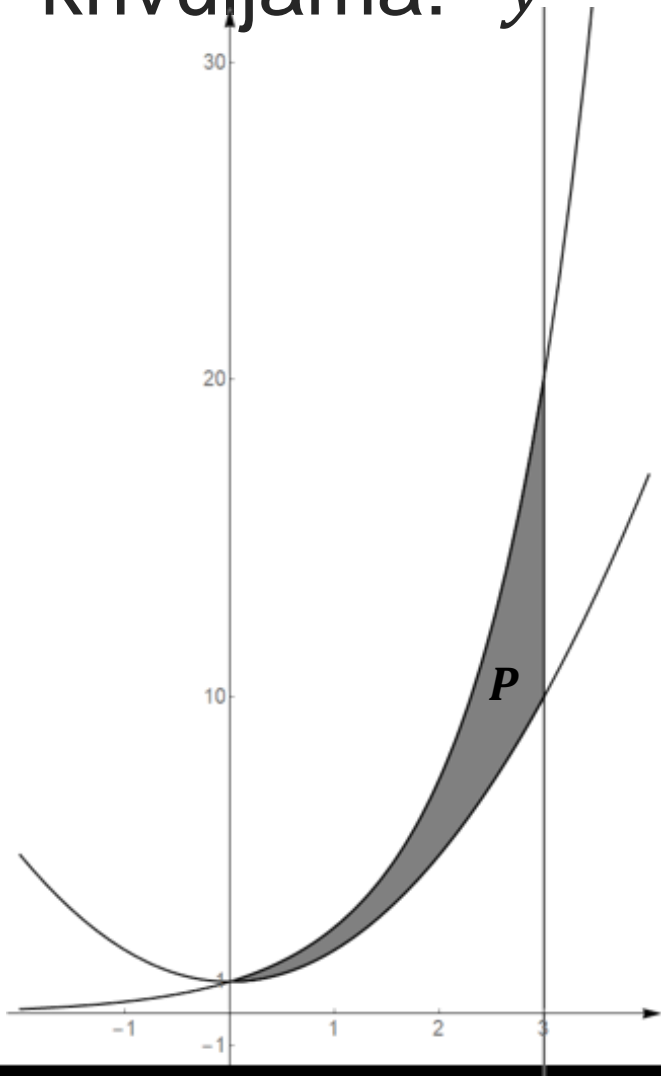


$$P_1 = \int_2^4 (x - (4 - x)) dx = 4$$

$$P_2 = \int_4^5 (x - (x^2 - 4x)) dx = \frac{13}{6}$$

$$P = P_1 + P_2 = \frac{37}{6}$$

Zadatak 6. Izračunajte površinu lika omeđenog krivuljama: $y = e^x$, $y = x^2 + 1$, $x = 3$.



$$\begin{aligned} P &= \int_0^3 (e^x - (x^2 + 1)) dx \\ &= \left(e^x - \frac{x^3}{3} - x \right) \Big|_0^3 \\ &= \left(e^3 - \frac{3^3}{3} - 3 \right) - \left(e^0 - \frac{0^3}{3} - 0 \right) = e^3 - 13 \end{aligned}$$

Hvala 😊