Određeni integral

NEWTON-LEIBNITZOVA FORMULA

$$\int_a^b f(x)\cdot dx = \underbrace{F(x)}_{\text{NEODREĐENI INTEGRAL}}\Big|_a^b = \underset{\text{gornja granica u sve x-eve}}{F(b)} - \underset{\text{donja u sve x-eve}}{F(a)}$$

TABLIČNI INTEGRALI

12.1. Izračunajte slijedeće integrale

a) $\int_0^1 (3x^2 - 4x + 2)\,dx = \left(\frac{\cancel{3}x^3}{\cancel{3}} - \cancel{4}^{2}\frac{x^2}{\cancel{2}} + 2x\right)\Big|_0^1 = 1^3 - 2\cdot 1^2 + 2\cdot 1 - \left(\cancel{0^3} - \cancel{2\cdot 0^2} + \cancel{2\cdot 0}\right)$

gornji u x —

$= 1 - 2 + 2$

$= 1$

b) $\int_0^2 x(3x+2)dx = \int_0^2 (3x^2 + 2x)\cdot dx = \left(\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_0^2 = 2^3 + 2^2 - (0^3 + 0^2)$

$= 8 + 4 = 12$

$\int \frac{1}{x}dx = \ln|x|$

c) $\int_1^2 \frac{x^2+1}{x}dx = \int_1^2 \left(\frac{x^{\cancel{2}}}{\cancel{x}} + \frac{1}{x}\right)\cdot dx = \int_1^2 \left(x + \frac{1}{x}\right)\cdot dx$

$= \left(\frac{x^2}{2} + \ln|x|\right)\Big|_1^2$

$= \underset{\text{GORNJA}}{\frac{2^2}{2} + \ln 2} - \underset{\text{DONJA}}{\left(\frac{1^2}{2} + \ln 1\right)}$

$\boxed{\ln 1 = 0}$

$= 2 + \ln 2 - \frac{1}{2} - 0$

$= \frac{3}{2} + \ln 2 \approx 2.1931$

SUPSTITUCIJA

$\int_1^2 \frac{x}{x^2+1}dx$

$\int A\cdot dx = Ax + c$

1 1 /5 3 \1 gornje − donje

$$\int A\cdot dx = Ax + c$$

d) $\displaystyle\int_0^1 (x^2+1)^2\,dx = \int (x^4+2x^2+1)\cdot dx = \left(\frac{x^5}{5} + \frac{2x^3}{3} + 1x\right)\Big|_0^1 = \frac{1}{5}+\frac{2}{3}+1-(0+0+0)$ gornje − donje

$$(a+b)^2 = a^2+2ab+b^2$$

$$= \frac{28}{15}$$

METODA SUPSTITUCIJE

12.2. Izračunajte slijedeće

a) $\displaystyle\int_2^3 \frac{3x^2}{x^3-1}\,dx = \ln|x^3-1|\Big|_2^3 = \ln|3^3-1| - \ln|2^3-1| = \ln 26 - \ln 7 \approx 1.3121$ gornje − donje, $\ln\frac{26}{7}$

$$\int \frac{1}{x}\cdot dx = \ln|x|$$

$$\int \frac{3x^2}{x^3-1}\boxed{dx} = \left|\begin{array}{l} x^3-1 = t \;/\,' \\ 3x^2\cdot dx = 1\cdot dt \;/ : 3x^2 \\ \boxed{dx} = \boxed{\dfrac{dt}{3x^2}} \end{array}\right| = \int \frac{\cancel{3x^2}}{t}\cdot\frac{dt}{\cancel{3x^2}} = \int \frac{1}{t}\cdot dt = \ln|t| + c$$

ZAMJENA

$$= \ln|x^3-1| + c$$

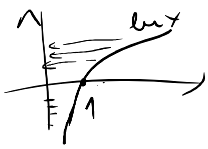
$$\frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}$$

b) $\displaystyle\int_0^6 (x-3)e^{x^2-6x}\,dx = \frac{1}{2}e^{x^2-6x}\Big|_0^6 = \frac{1}{2}\cdot\left(e^{6^2-6\cdot 6} - e^{0^2-0}\right) = \frac{1}{2}(1-1) = \frac{1}{2}\cdot 0 = 0$

(e^0)

$$\int (x-3)\cdot e^{x^2-6x}\cdot\boxed{dx} = \left|\begin{array}{l} x^2-6x = t \\ (2x-6)\cdot dx = dt \;/ : (\;\;) \\ \boxed{dx} = \dfrac{dt}{2x-6} = \boxed{\dfrac{dt}{2(x-3)}} \end{array}\right| = \int \cancel{(x-3)}\cdot e^t\cdot\frac{dt\cdot 1}{2\cancel{(x-3)}} = \frac{1}{2}\int e^t\cdot dt$$

$$= \frac{1}{2}e^t + c = \frac{1}{2}e^{x^2-6x} + c$$



c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\ln|\cos x|\Big|_0^{\pi/4} = -\ln\Big|\underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\Big| - \Big(-\ln|\underbrace{\cos 0}_{1}|\Big) = -\ln\frac{\sqrt{2}}{2} + \underbrace{\ln 1}_{0} = -\ln\frac{\sqrt{2}}{2} \approx 0.3466$

"tg x"; gornje − donja

$0 \qquad 0 \qquad \frac{\sqrt{2}}{2} \qquad 1$

$$\int \frac{\sin x}{\cos x}\cdot dx = \left| \begin{array}{l} \cos x = t \ /' \\ -\sin x\cdot dx = dt \\ dx = \frac{dt}{-\sin x} \end{array} \right| = \int \frac{\cancel{\sin x}}{t}\cdot\frac{dt}{-\cancel{\sin x}} = -\int \frac{1}{t}\cdot dt = -\ln|t| + c$$

$$= -\ln|\cos x| + c$$

$\ln_e e = 1$ ezsponent

$\ln_e e^2 = 2$

$\ln_e e^3 = 3 \quad \ldots$

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d) $$\int_e^{e^2} \frac{dx}{x\ln x} = \ln|\ln x|\Big|_e^{e^2} = \ln|\underbrace{\ln e^2}_{2}| - \ln|\underbrace{\ln e}_{1}| = \ln 2 - \underbrace{\ln 1}_{0} = \ln 2$$ //

gornje − donje

$$\int \frac{\boxed{dx}}{x \ln x} = \left| \begin{array}{l} \ln x = t \ /' \\ \frac{1}{x}\cdot dx = dt \ /\cdot x \\ \boxed{dx} = \boxed{x\cdot dt} \end{array} \right| = \int \frac{\cancel{x}\cdot dt}{\cancel{x}\cdot t} = \int \frac{1}{t}\, dt = \ln|t| + c = \ln|\ln x| + c$$

PARCIJALNA INTEGRACIJA

12.3. Izračunajte slijedeće :

a) $$\int_0^{\pi/2} x\cos x\, dx = (x\sin x + \cos x)\Big|_0^{\pi/2} = \frac{\pi}{2}\underbrace{\sin\frac{\pi}{2}}_{1} + \underbrace{\cos\frac{\pi}{2}}_{0} - \left(\cancel{0\sin 0} + \underbrace{\cos 0}_{1}\right) = \frac{\pi}{2} - 1$$ //

gornje − donje

$$\int u\cdot dv = uv - \int v\cdot du$$

$$\int x\cdot\cos x\cdot dx = \left| \begin{array}{lcl} u = x & \xrightarrow{\text{DER.}} & 1\cdot du = 1\cdot dx \\ dv = \cos x\cdot dx & \xrightarrow{\text{INT.}} & v = \sin x \end{array} \right|$$

$$= x\cdot\sin x - \int \sin x\cdot dx$$

$$= x\sin x + \cos x + c$$

1. TIP : $\int \text{POLINOM} \cdot \left\{ \begin{array}{c} \sin(ax+b) \\ \text{ili} \\ \cos(ax+b) \\ \text{ili} \\ e^{ax+b} \end{array} \right\} \cdot dx$

POLINOM: $x, x^2, x+1, x^2-2, \ldots \rightarrow u$; $\sin(ax+b)$ ili $\cos(ax+b)$ ili $e^{ax+b} \rightarrow dv$

c) $\int_0^1 x e^{2x} dx = (*)$

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$$\int x e^{2x} \cdot dx = \left| \begin{array}{lcl} u = x & \xrightarrow{der.} & du = dx \\ dv = e^{2x} \cdot dx & \xrightarrow{int.} & v = \frac{1}{2} e^{2x} \end{array} \right|$$

1. TIP: $\int \left\{ \begin{array}{c} \text{POLINOM} \\ x, x^2, x+1, \\ x^2-2, \dots \end{array} \right\} \cdot \left\{ \begin{array}{c} \sin(ax+b) \\ \text{ili} \\ \cos(ax+b) \\ \text{ili} \\ e^{ax+b} \end{array} \right\} \cdot dx$, POLINOM $\rightarrow u$, $\sin/\cos/e \rightarrow dv$

MOŽEMO NAPAMET:

$\int e^{ax+b} \cdot dx = \frac{1}{a} \cdot e^{ax+b} + c$

$\int \cos(ax+b) \cdot dx = \frac{1}{a} \sin(ax+b) + C$

$\int \sin(ax+b) \cdot dx = -\frac{1}{a} \cos(ax+b) + C$

$$= x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \cdot dx = \frac{1}{2} x e^{2x} - \frac{1}{2} \int \overbrace{e^{2x} \cdot dx}^{\frac{1}{2} e^{2x}} = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

$$(*) = \left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right) \Big|_0^1 = \underbrace{\frac{1}{2} \cdot 1 \cdot e^2 - \frac{1}{4} e^2}_{\text{gornja}} - \underbrace{\left(\cancel{\frac{1}{2} \cdot 0 \cdot e^0} - \frac{1}{4} e^0 \right)}_{\text{donja}} = \frac{1}{4} e^2 + \frac{1}{4}$$

$\boxed{e^0 = 1}$

b) $\int_1^2 3x^2 \ln x \, dx = \left(x^3 \ln x - \frac{x^3}{3} \right) \Big|_1^2 = 2^3 \ln 2 - \frac{2^3}{3} - \left(1 \underbrace{\ln 1}_{0} - \frac{1}{3} \right) = 8 \ln 2 - \frac{8}{3} + \frac{1}{3} = 8 \ln 2 - \frac{7}{3}$

$$\int 3x^2 \ln x \cdot dx = \left| \begin{array}{lcl} u = \ln x & \xrightarrow{der.} & du = \frac{1}{x} \cdot dx \\ dv = 3x^2 \cdot dx & \xrightarrow{int.} & v = \frac{\cancel{3} x^3}{\cancel{3}} = x^3 \end{array} \right|$$

2. TIP: $\int \left\{ \begin{array}{c} \text{POLINOM} \\ \text{ili} \\ \text{POTENCIJA} \end{array} \right\} \cdot \ln x \cdot dx$, $x, x^2, \sqrt{x}, \frac{1}{x^2}, \dots, x+1 \rightarrow dv$, $\ln x \rightarrow u$

$= \ln x \cdot x^3 - \int x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}} \cdot dx$

$= x^3 \cdot \ln x - \int x^2 \cdot dx$

$= x^3 \cdot \ln x - \frac{x^3}{3} + c$