# MATEMATIČKA ANALIZA

## Limes funkcije

# Zenonov paradoks

Ahilej nikada ne stiže kornjaču:

- kada Ahilej stigne do mjesta gdje je kornjača bila, ona odmakne malo dalje

# Zenonov paradoks

Ako Ahilej prvi puta potroši jednu minutu da dođe do mjesta gdje je bila kornjača.

Drugi puta pola minute do mjesta gdje je bila kornjača.

Potom četvrtinu minute, itd.

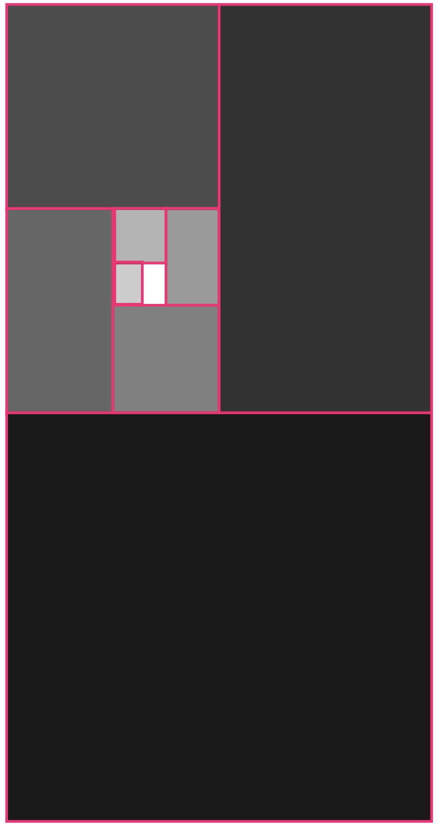
$$t = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

# Zenonov paradoks

Možemo vidjeti kako taj izraz ***teži*** ka nekoj vrijednosti.

U ovom konkretnom slučaju, prikazana suma teži ka 2.

$$t = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$



# Limes funkcije

**Intuitivna definicija limesa funkcije:**

Ako se vrijednost funkcije $f(x)$ približava vrijednosti $L \in \mathbb{R}$, kad se nezavisna varijabla $x$ približava vrijednosti $c$, tada kažemo da $f(x)$ teži prema $L \in \mathbb{R}$ kada $x$ teži prema $c$ i pišemo:

$$f(x) \to L \quad \text{kada} \quad x \to c$$

$$\lim_{x \to c} f(x) = L$$

# Limes funkcije

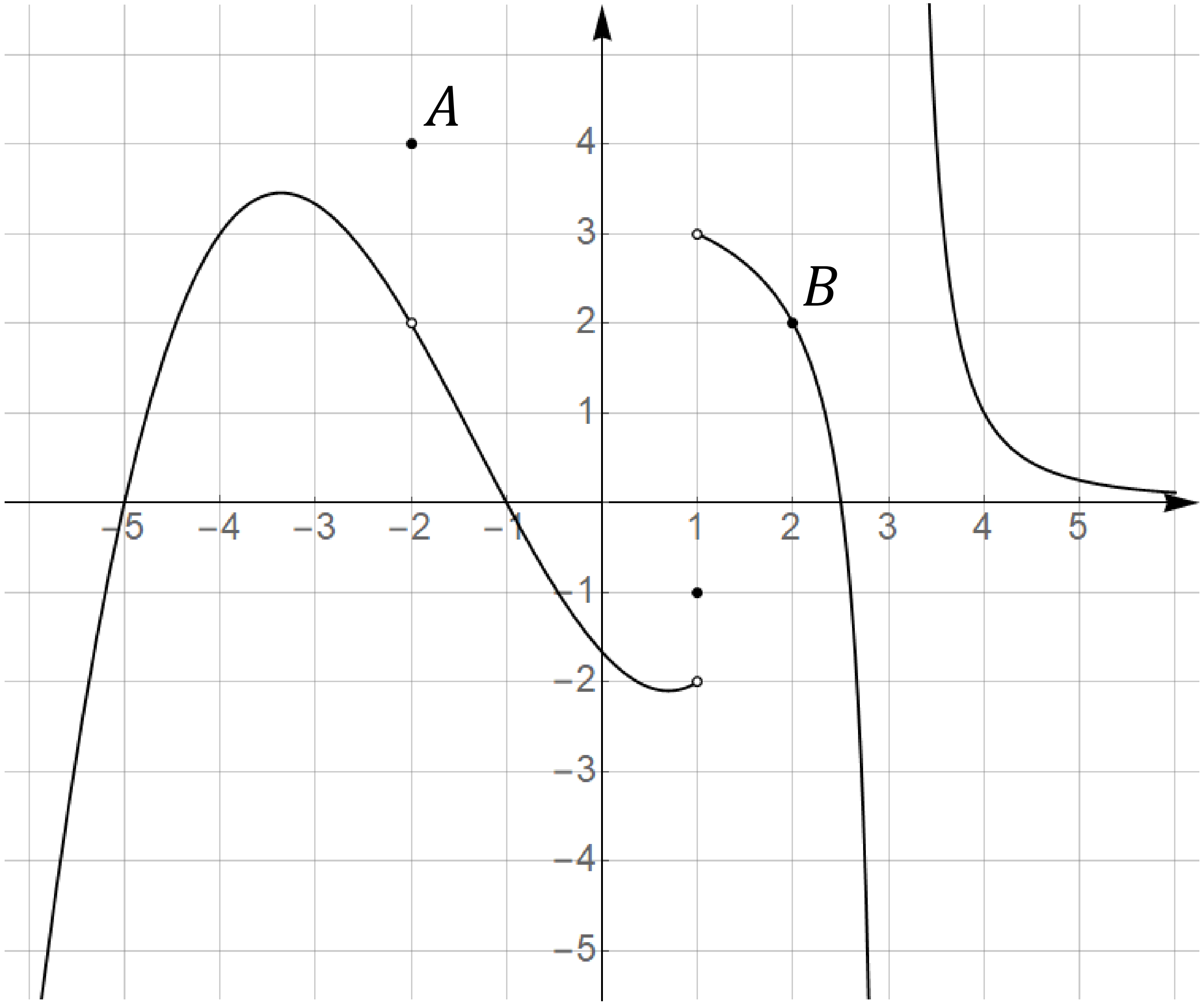
Funkcija $f(x)$ ima limes $L \in \mathbb{R}$ u točki $c \in \mathbb{R}$ ako vrijedi

$$(\forall \varepsilon > 0)(\exists \delta > 0)\ \forall x \in D_f \backslash \{c\}$$

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Ako postoji takav $L \in \mathbb{R}$ kažemo da funkcija $f(x)$ **konvergira** u točki $c$. Ako limes ne postoji kažemo da funkcija **divergira** u točki $c$.

# Limes funkcije




Funkcija ima prekid za $x = -2$.

$f(-2) = 4$ $\lim_{x \to -2} f(x) = 2$

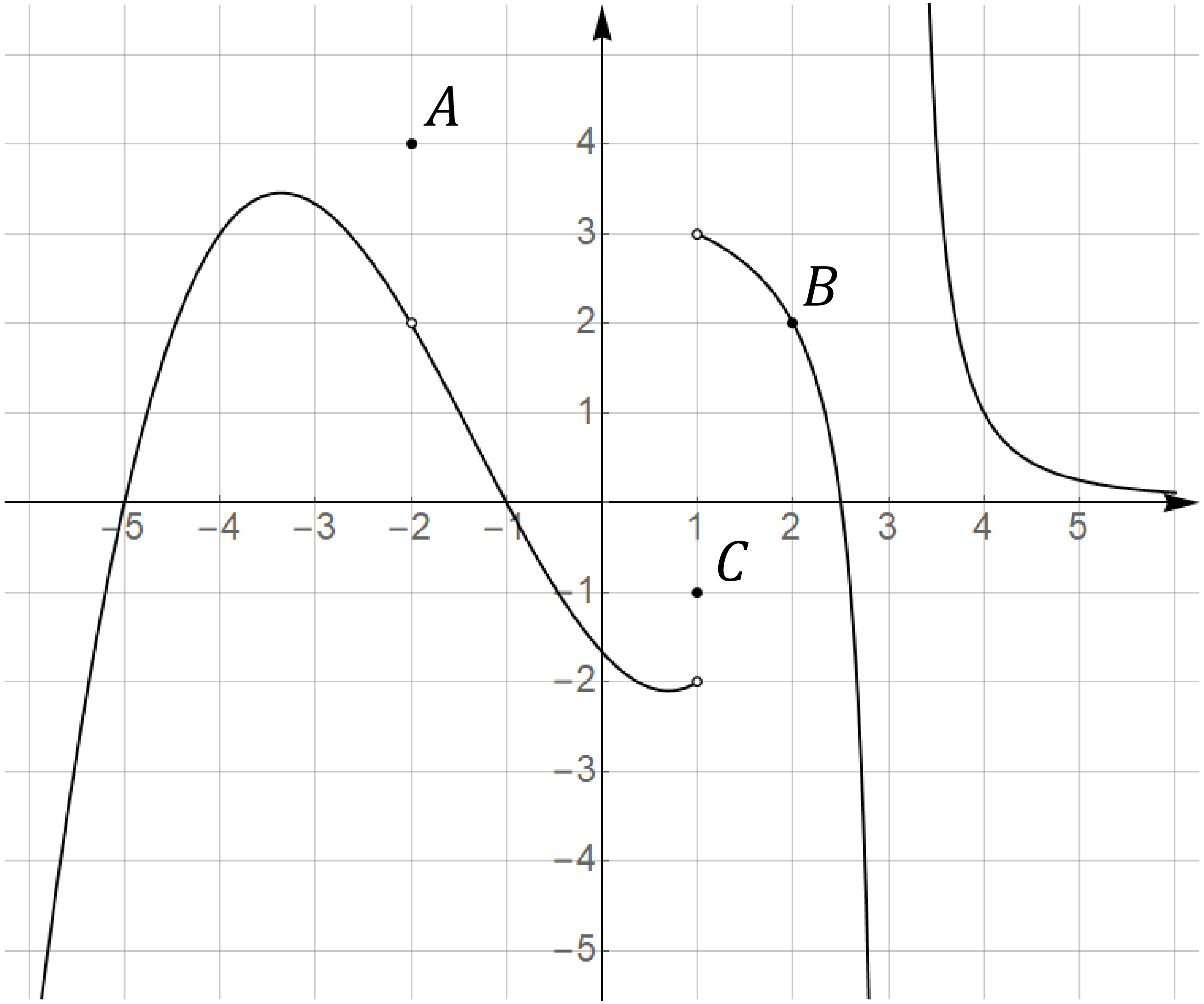
Točka $A$.

$f(2) = 2$ $\lim_{x \to 2} f(x) = 2$

Točka $B$.

U $x = 2$ funkcija je neprekidna.

# Limes funkcije




Funkcija ima prekid za $x = 1$.

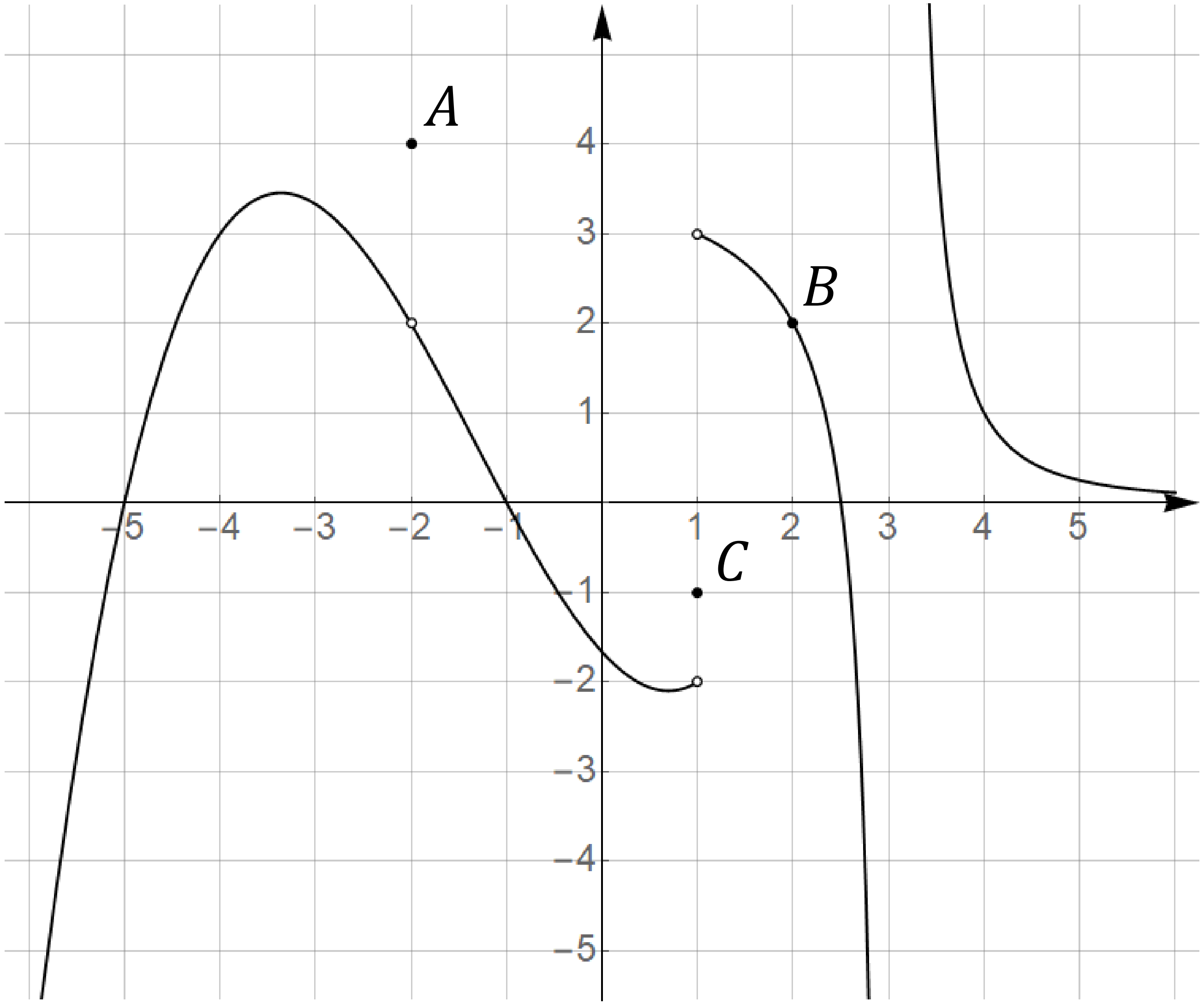
$f(1) = -1$ $\lim_{x \to 1^-} f(x) = -2$

Točka $C$.

U ovom slučaju razlikujemo **lijevi** i **desni** limes!

**Lijevi** limes je kada se točki približavamo s **negativne** strane.

# Limes funkcije




Funkcija ima prekid za $x = 1$.

$f(1) = -1$

$\lim\limits_{x \to 1^-} f(x) = -2$

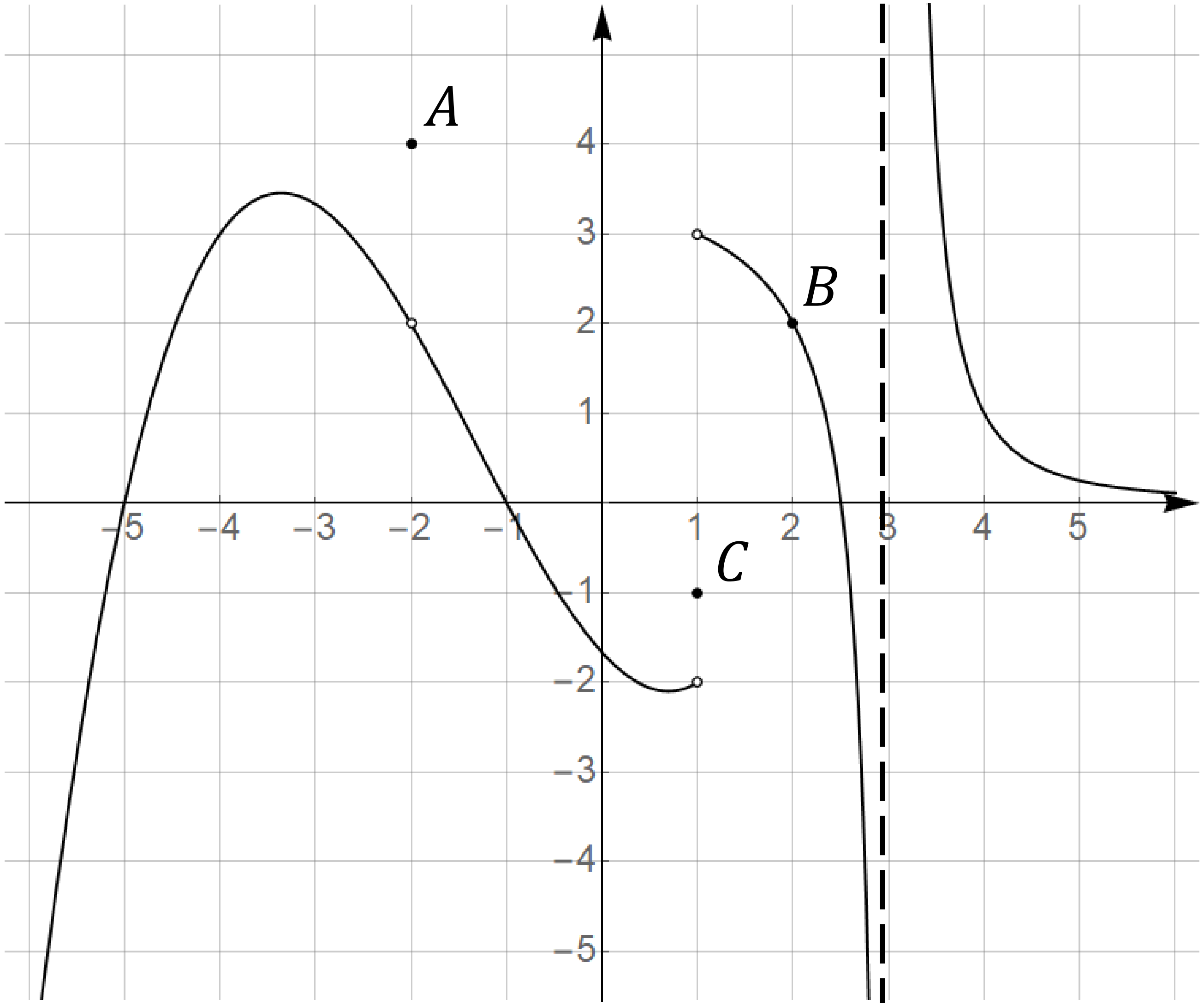
Točka $C$.

$\lim\limits_{x \to 1^+} f(x) = 3$

**Desni** limes je kada se točki približavamo s **pozitivne** strane.

$\lim\limits_{x \to 1} f(x)$ **ne postoji!**

# Limes funkcije




Funkcija ima prekid i za $x = 3$.

Za taj $x$ funkcija nije definirana.

$$\lim_{x \to 3^-} f(x) = -\infty$$

$$\lim_{x \to 3^+} f(x) = \infty$$

$$\lim_{x \to 3} f(x) = \pm\infty$$

# Limes funkcije

Funkcija $f(x)$ ima limes $L \in \mathbb{R}$ u beskonačnosti ako vrijedi:

$$(\forall \varepsilon > 0)(\exists M > 0), \forall x > M \ \Rightarrow \ |f(x) - L| < \varepsilon$$

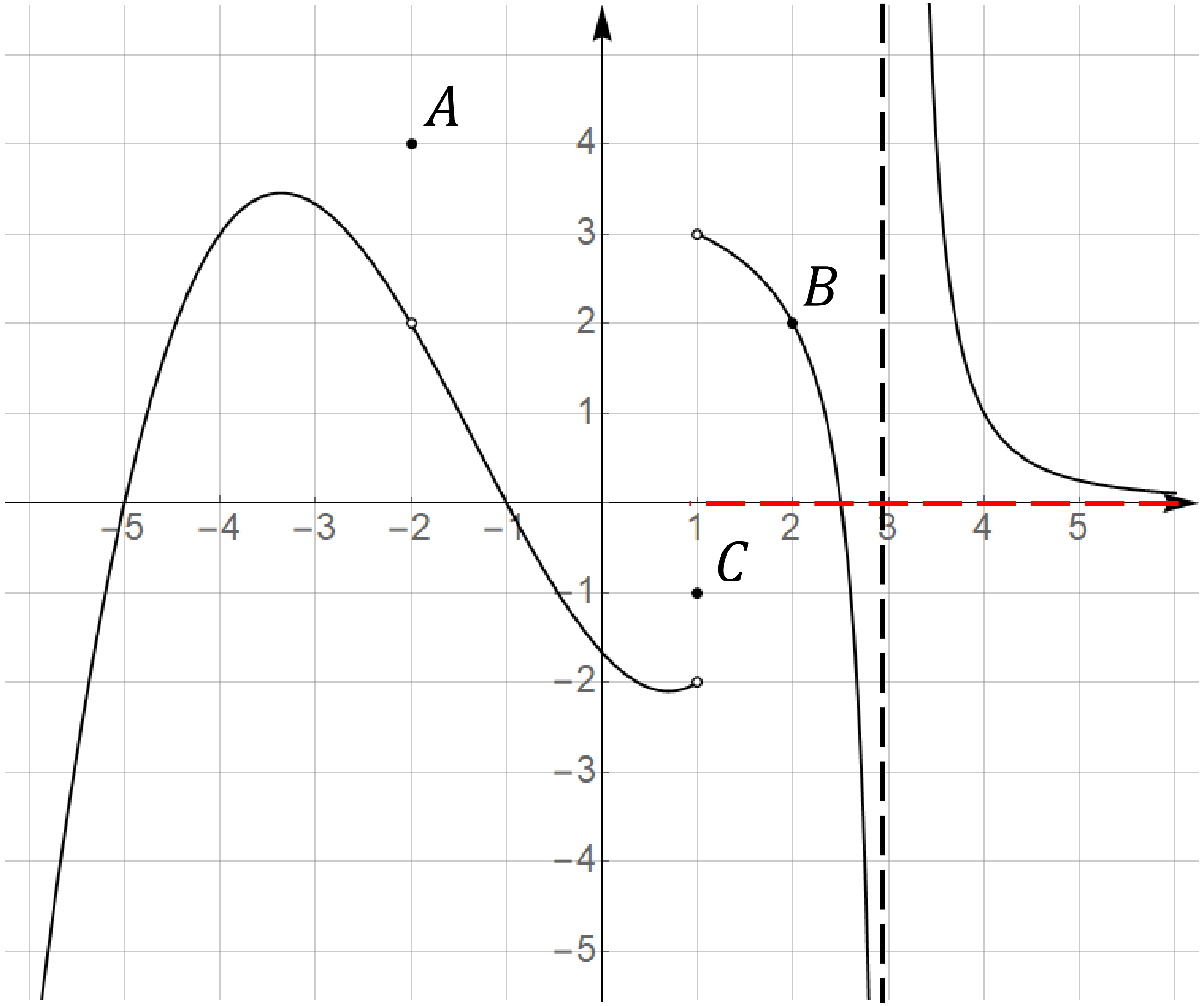
$$\lim_{x \to \infty} f(x) = L$$

Funkcija $f(x)$ ima limes $L \in \mathbb{R}$ u negativnoj beskonačnosti ako vrijedi:

$$(\forall \varepsilon > 0)(\exists M < 0), \forall x < M \ \Rightarrow \ |f(x) - L| < \varepsilon$$

$$\lim_{x \to -\infty} f(x) = L$$

# Limes funkcije

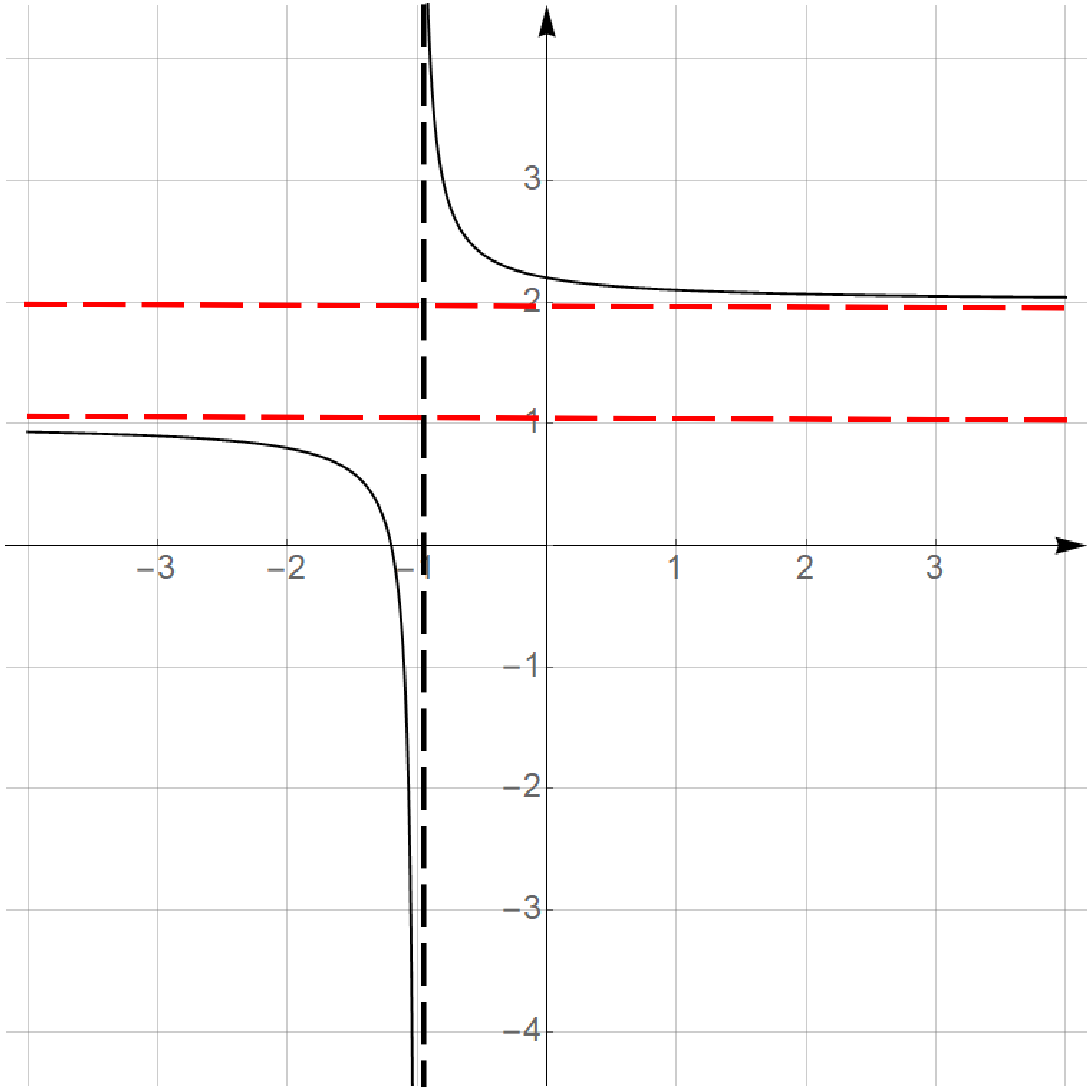



Za limes u (negativnoj) beskonačnosti, gledamo kamo se približava vrijednost funkcije $y$ kada $x$ neograničeno raste na desnu (lijevu) stranu.

$$\lim_{x \to \infty} f(x) = 0$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

# Limes funkcije




$$\lim_{x \to -1^-} f(x) = -\infty$$

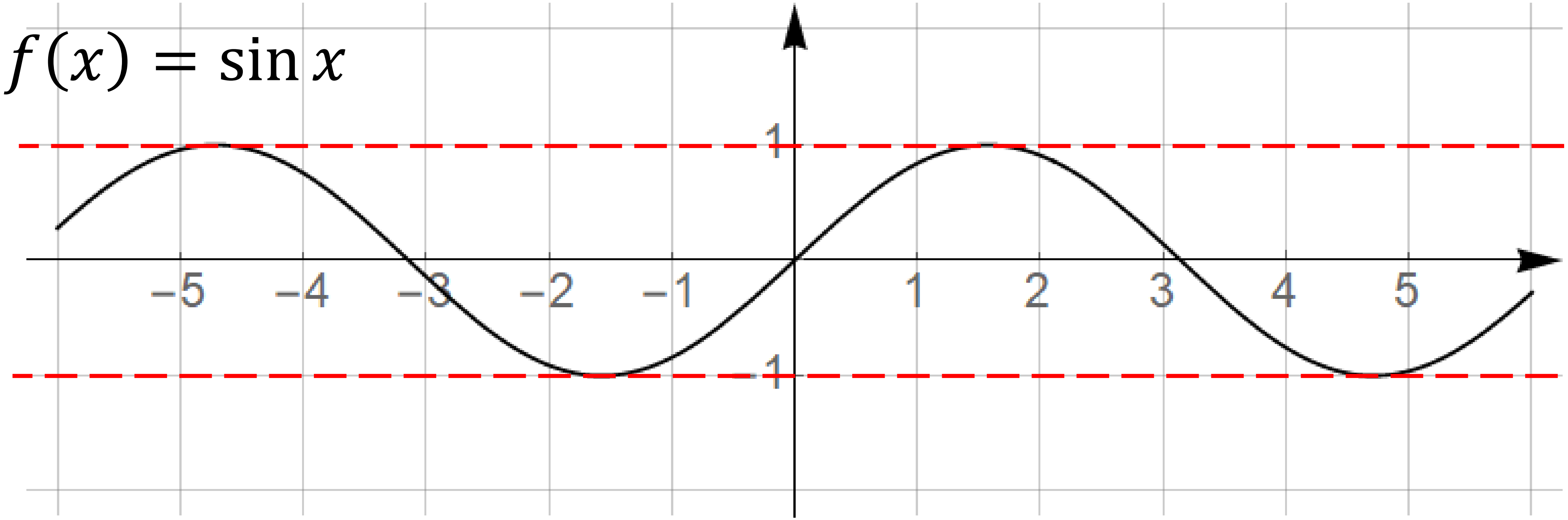
$$\lim_{x \to -1^+} f(x) = \infty$$

$$\lim_{x \to \infty} f(x) = 2$$

$$\lim_{x \to -\infty} f(x) = 1$$

# Limes funkcije

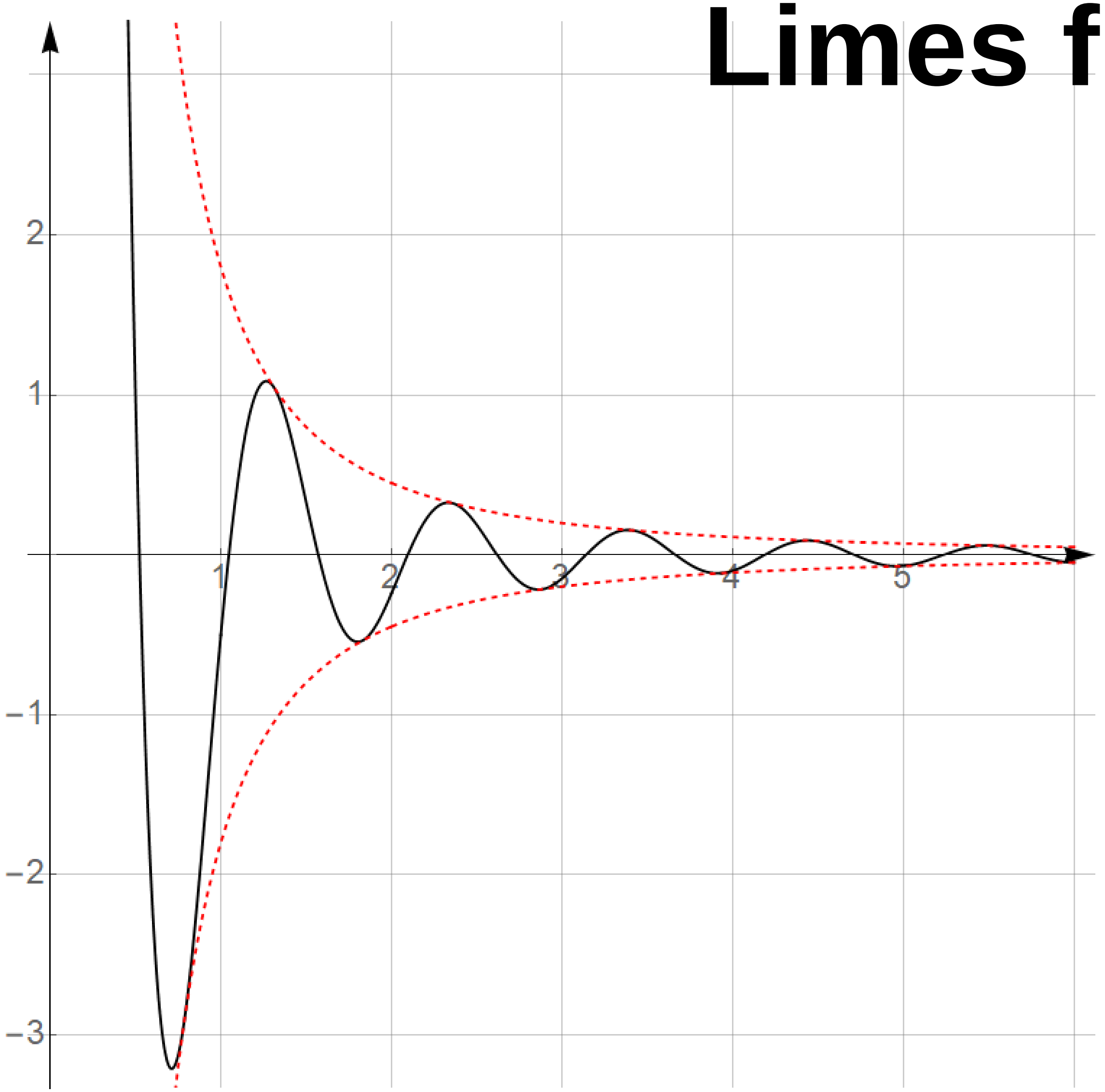
$f(x) = \sin x$




$\lim_{x \to -\infty} f(x)$ ne postoji!

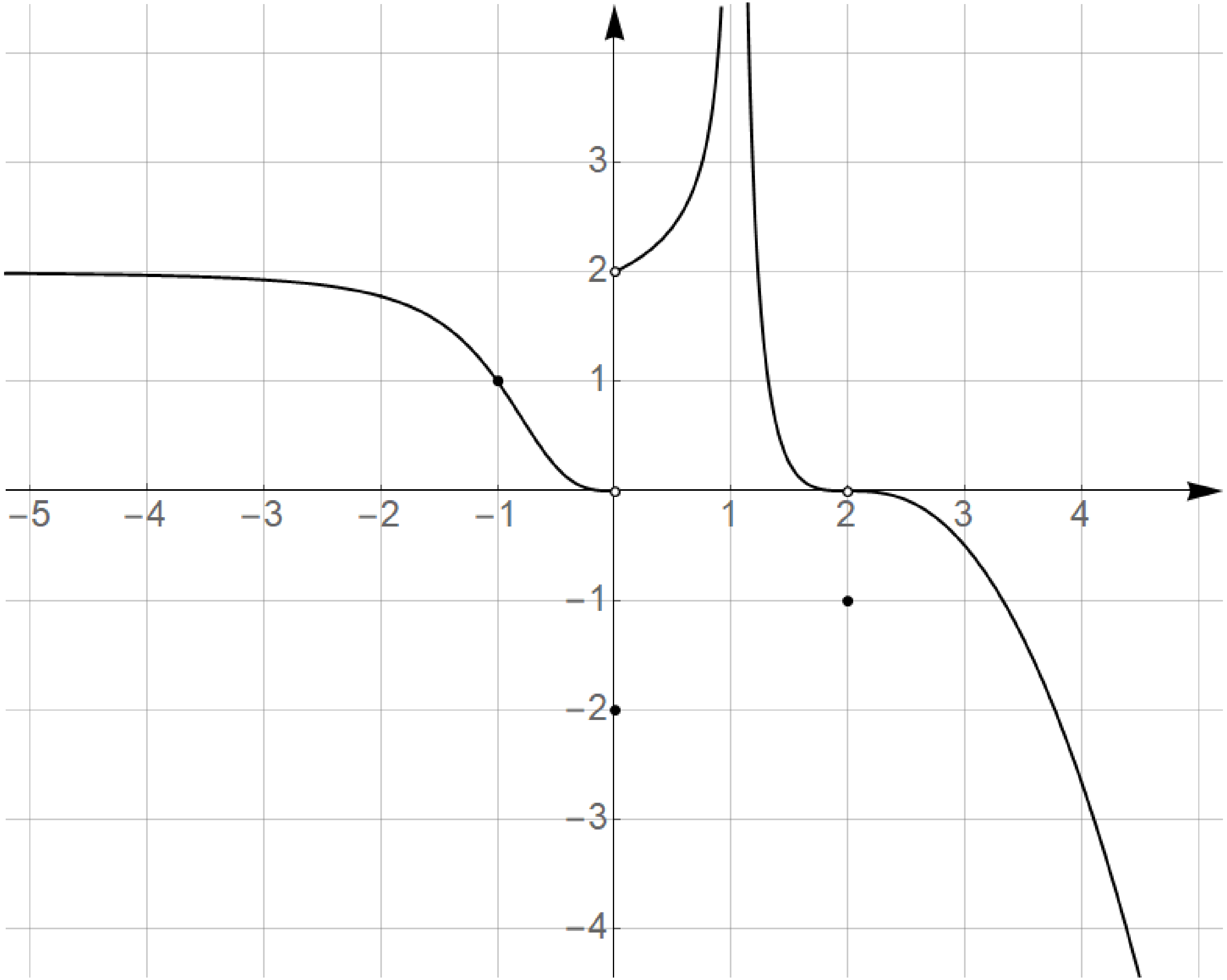
$\lim_{x \to \infty} f(x)$ ne postoji!

# Limes funkcije




$$\lim_{x \to \infty} f(x) = 0$$

$$f(x) = \frac{1.8}{x^2} \sin(6x)$$




$$\lim_{x\to -1} f(x) = 1$$

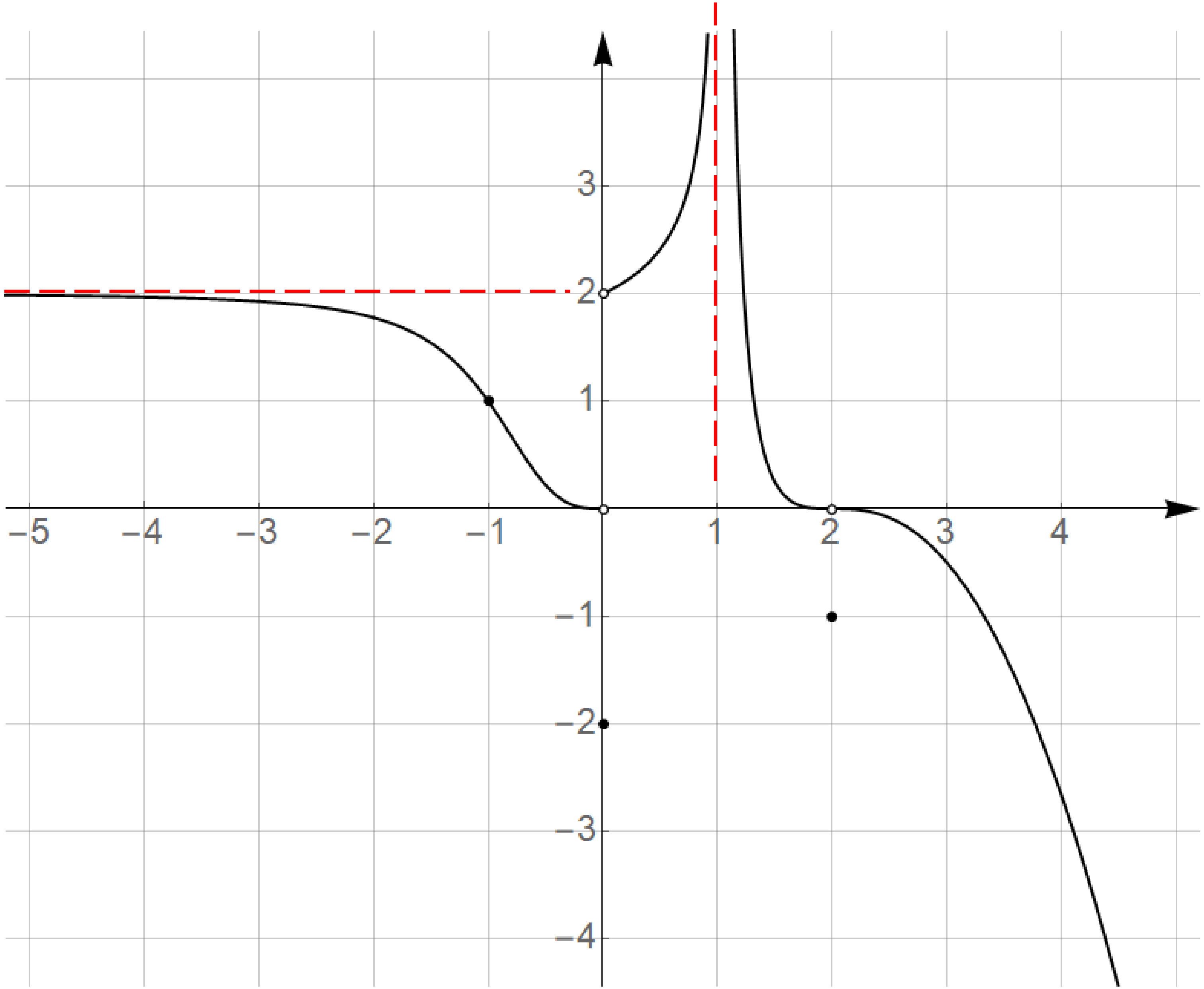
$$\lim_{x\to 0} f(x) =$$ ne postoji

$$\lim_{x\to 0^-} f(x) = 0$$

$$\lim_{x\to 0^+} f(x) = 2$$

$$\lim_{x\to 2} f(x) = 0$$

$$f(2) = -1$$




$$\lim_{x \to -\infty} f(x) = 2$$

$$\lim_{x \to \infty} f(x) = -\infty$$

$$\lim_{x \to 1^-} f(x) = \infty$$

$$\lim_{x \to 1^+} f(x) = \infty$$

$$f(-1) = 1$$

$$f(0) = -2$$

# Limes funkcije

## Određeni oblici

$$\frac{0}{\infty} = 0 \qquad \frac{\infty}{0} = \infty$$

$$\frac{a}{\infty} = 0 \qquad \frac{\infty}{a} = \infty$$

$$\frac{0}{a} = 0 \qquad \frac{a}{0} = \infty$$

## Neodređeni oblici

$$\frac{\infty}{\infty} \qquad \frac{0}{0}$$

$$\infty - \infty \qquad 1^{\infty}$$

# Računanje limesa

Odredite limese funkcija:

a) $\lim_{x\to 7} \frac{1}{x-8} = \frac{1}{-1} = -1$

b) $\lim_{x\to 8} \frac{1}{x-8} = \left[\frac{1}{0}\right] = \infty$

$\lim_{x\to 8^+} \frac{1}{x-8} = \left[\frac{1}{0^+}\right] = +\infty$

$\lim_{x\to 8^-} \frac{1}{x-8} = \left[\frac{1}{0^-}\right] = -\infty$

c) $\lim_{x\to \infty} \frac{1}{x-8} = \left[\frac{1}{\infty}\right] = 0$

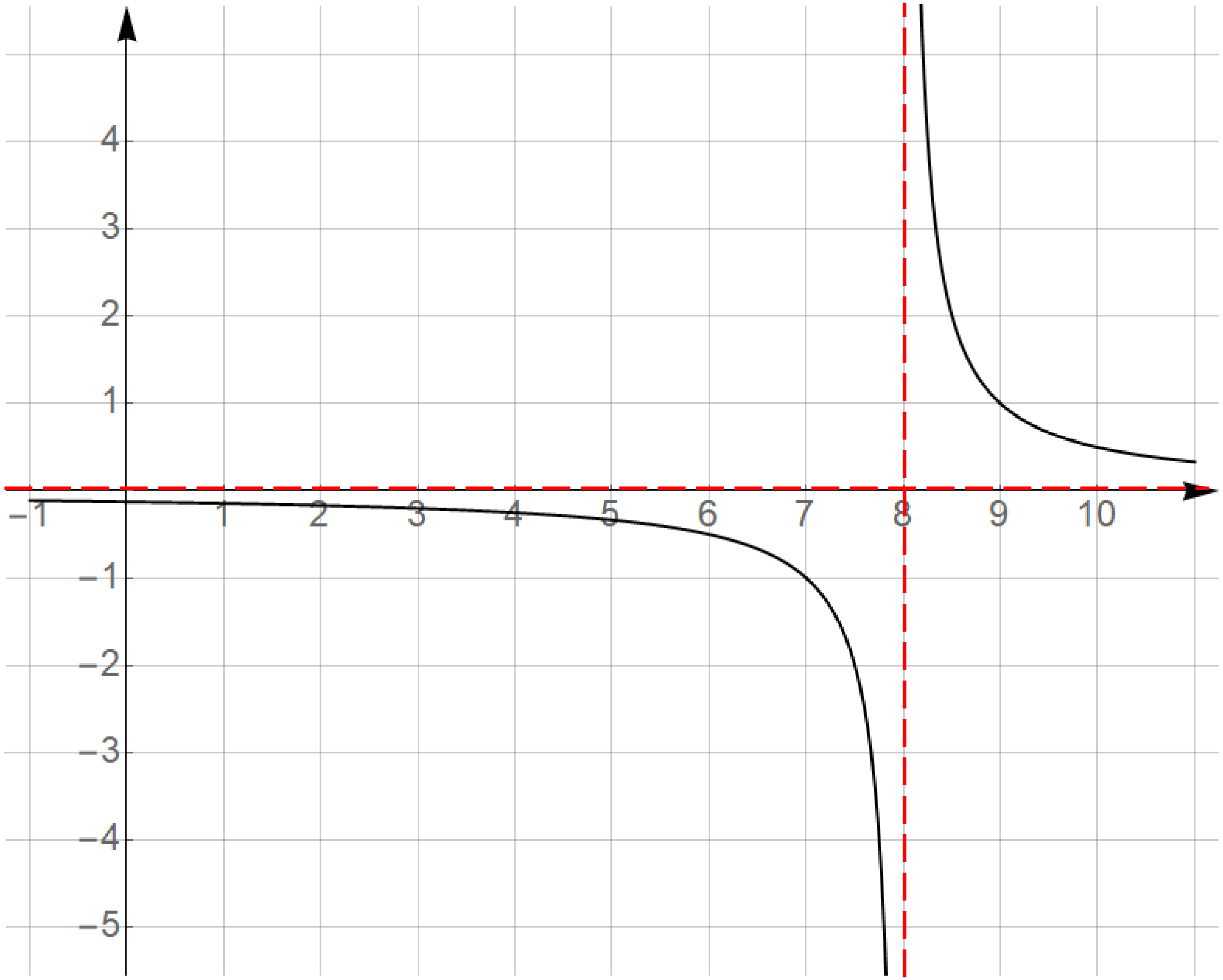
# Grafički prikaz funkcije

$$f(x) = \frac{1}{x-8}$$

$$\lim_{x \to 8^+} \frac{1}{x-8} = \infty$$

$$\lim_{x \to 8^-} \frac{1}{x-8} = -\infty$$

$$\lim_{x \to \infty} \frac{1}{x-8} = 0$$



After explaining to a student through various lessons and examples that:

$$\lim_{x\to 8}\frac{1}{x-8} = \infty$$

I tried to check if he really understood that, so I gave him a different example.

This was the result:

$$\lim_{x\to 5}\frac{1}{x-5} = \rotatebox{90}{5}$$

# Računanje limesa

Odredite limese funkcija:

d) $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = 6$

e) $\lim\limits_{x\to -5} \frac{x^2-2x-15}{x^2+5x} = \left[\frac{20}{0}\right] = \infty$

f) $\lim\limits_{x\to 5} \frac{x^2-2x-15}{x^2-5x} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 5} \frac{\cancel{(x-5)}(x+3)}{x\cancel{(x-5)}} = \frac{8}{5}$

# Računanje limesa

Odredite limese funkcija:

g) $$\lim_{x\to 8}\frac{2-\sqrt{x-4}}{x-8}\cdot\frac{2+\sqrt{x-4}}{2+\sqrt{x-4}} = \lim_{x\to 8}\frac{2^2-\left(\sqrt{x-4}\right)^2}{(x-8)(2+\sqrt{x-4})}$$

$$= \lim_{x\to 8}\frac{8-x}{(x-8)(2+\sqrt{x-4})} = \lim_{x\to 8}\frac{-1}{2+\sqrt{x-4}} = -\frac{1}{4}$$

h) $$\lim_{x\to 1}\frac{x^2-1}{3-\sqrt{10-x}}\cdot\frac{3+\sqrt{10-x}}{3+\sqrt{10-x}} = \lim_{x\to 1}\frac{(x^2-1)\left(3+\sqrt{10-x}\right)}{3^2-\left(\sqrt{10-x}\right)^2}$$

$$= \lim_{x\to 1}\frac{(x-1)(x+1)\left(3+\sqrt{10-x}\right)}{x-1} = 2\cdot 6 = 12$$

# Računanje limesa

Odredite limese funkcija:

i) $$\lim_{x\to\infty}\frac{x^2-2x-15}{x^2+5x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{x^2-2x-15}{x^2+5x}\,\frac{:x^2}{:x^2} =$$

$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{15}{x^2}}{\frac{x^2}{x^2}+\frac{5x}{x^2}} = \lim_{x\to\infty}\frac{1-\frac{2}{x}-\frac{15}{x^2}}{1+\frac{5}{x}} = 1$$

# Računanje limesa

Odredite limese funkcija:

j) $\lim\limits_{x\to\infty} \frac{2-\sqrt{x^2-4}}{x-8} = \left[\frac{-\infty}{\infty}\right] = \lim\limits_{x\to\infty} \frac{2-\sqrt{x^2-4}}{x-8} \frac{:x}{:x} =$

$$= \lim_{x\to\infty} \frac{\frac{2}{x} - \frac{\sqrt{x^2-4}}{x}}{\frac{x}{x} - \frac{8}{x}} = \lim_{x\to\infty} \frac{\frac{2}{x} - \sqrt{\frac{x^2-4}{x^2}}}{1 - \frac{8}{x}} = \lim_{x\to\infty} \frac{\frac{2}{x} - \sqrt{1-\frac{4}{x^2}}}{1 - \frac{8}{x}} = -1$$

# Računanje limesa

Odredite limese funkcija:

k) $\lim\limits_{x\to\infty}\left(\sqrt{x-3}-\sqrt{x+2}\right) = [\infty-\infty] = \left(\cdot\frac{\sqrt{x-3}+\sqrt{x+2}}{\sqrt{x-3}+\sqrt{x+2}}\right) =$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x-3}\right)^2-\left(\sqrt{x+2}\right)^2}{\sqrt{x-3}+\sqrt{x+2}} = \lim_{x\to\infty}\frac{x-3-(x+2)}{\sqrt{x-3}+\sqrt{x+2}}$$

$$= \lim_{x\to\infty}\frac{-5}{\sqrt{x-3}+\sqrt{x+2}} = \left[\frac{-5}{\infty}\right] = 0$$

# Video materijali Tonija Miluna

https://www.youtube.com/watch?v=vaJhttcgz2s&list=PL3E167BBDE2E5BCF4

Hvala ☺