Vježbe 11

14. svibnja 2024. 11:15

METODA PARCIJALNE INTEGRACIJE

$$\boxed{\int u \cdot dv = u \cdot v - \int v \cdot du}$$

1. TIP:

$$\int \begin{matrix} \text{POLINOM} \\ x,\ x^2,\ x+1,\ x^2-2,\ \dots \\ \downarrow \\ u \end{matrix} \cdot \left\{ \begin{matrix} \sin(ax+b) \\ \text{ili} \\ \cos(ax+b) \\ \text{ili} \\ e^{ax+b} \end{matrix} \right\} \cdot dx \qquad \left\{\ldots\right\}dx \to dv$$

$$\int 5\, dx = 5x$$

$$\int 1 dx = 1x$$

$$\int dv = 1v$$

11.1

a) $\int x \sin x \, dx = \left| \begin{matrix} u = x & \xrightarrow{\text{DERIV.}} & 1 \cdot du = 1 \cdot dx \\ dv = \sin x \cdot dx & \xrightarrow{\text{INTEG.}} & v = -\cos x \end{matrix} \right|$

$= x \cdot (-\cos x) - \int -\cos x \cdot dx$ $\qquad \int \cos x \cdot dx = \sin x$

$= -x\cos x + \sin x + C$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

b) $\int x^2 \cos x \, dx = \left| \begin{matrix} u = x^2 & \xrightarrow{\text{DER.}} & du = 2x \cdot dx \\ dv = \cos x \cdot dx & \xrightarrow{\text{INT.}} & v = \sin x \end{matrix} \right|$

$= x^2 \sin x - \int \sin x \cdot 2x \cdot dx$

$= x^2 \sin x - 2 \cdot \int x \cdot \sin x \cdot dx = (*)$ $\quad \to \cancel{\frac{x^2}{2} \cdot (-\cos x)}$

OPET PARCIJALNA INT.:

$\int x \sin x \cdot dx = \left| \begin{matrix} u = x & \xrightarrow{\text{DER.}} & du = dx \\ dv = \sin x \cdot dx & \xrightarrow{\text{INT.}} & v = -\cos x \end{matrix} \right| = -x\cos x - \int -\cos x \cdot dx = -x\cos x + \sin x$

$(*) = x^2 \sin x - 2 \cdot (-x\cos x + \sin x) + C$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

c) $\int x e^{2x}\,dx = \left| \begin{array}{ll} u = x \xrightarrow{\text{DER.}} & du = dx \\ dv = e^{2x}\cdot dx \xrightarrow{\text{INT.}} & v = \int e^{2x} dx = \left| \begin{array}{l} 2x = t \,/' \\ 2\cdot dx = dt \;/:2 \\ dx = \frac{dt}{2} \end{array} \right| = \int e^t \cdot \frac{dt}{2} = \frac{1}{2}\int e^t \cdot dt \\ & \text{SUPSTITUCIJA} \\ & v = \frac{1}{2} e^{t} = \frac{1}{2} e^{2x} \end{array} \right|$

$= x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \cdot dx$

$= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} \cdot dx$

$= \frac{1}{2} x e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$

$\int u \cdot dv = u \cdot v - \int v \cdot du$

d) $\int \frac{x}{e^x}\,dx = \int x \cdot e^{-x} \cdot dx = \left| \begin{array}{ll} u = x \xrightarrow{\text{der.}} & du = dx \\ dv = e^{-x} dx \xrightarrow{\text{int.}} & v = \frac{1}{-1} e^{-x} = -e^{-x} \end{array} \right|$

$= x \cdot (-e^{-x}) - \int -e^{-x} \cdot dx$

$= -x e^{-x} + \underbrace{\int e^{-x} \cdot dx}$

$= -x e^{-x} - e^{-x} + C$

MOŽEMO NAPAMET:

$\int e^{ax+b} \cdot dx = \frac{1}{a} \cdot e^{ax+b} + c$

$\int \cos(ax+b) \cdot dx = \frac{1}{a} \sin(ax+b) + C$

$\int \sin(ax+b) \cdot dx = -\frac{1}{a} \cos(ax+b) + C$

upr. $\int e^{5x-7} dx = \frac{1}{5} e^{5x-7} + c$

$\int \cos(4x-8) dx = \frac{1}{4} \sin(4x-8) + c$

$\int \sin(6x-1) \cdot dx = -\frac{1}{6} \cos(6x-1) + c$

2. TIP:

$\int \left\{ \begin{array}{c} \text{POLINOM} \\ \text{ili} \\ \text{POTENCIJA} \end{array} \right\} \cdot \ln x \cdot dx$

$x, x^2, \sqrt{x}, \frac{1}{x^2}, \ldots$

$x+1$

$\downarrow$ dv $\qquad \downarrow$ u

$(\ln x)' = \frac{1}{x}$

11.2.

a) $\int 4x \ln x\,dx = \left| \begin{array}{ll} u = \ln x \xrightarrow{\text{DER.}} & du = \frac{1}{x} \cdot dx \\ dv = 4x^1 \cdot dx \xrightarrow{\text{INT.}} & v = \frac{^{2}\cancel{4} x^2}{\cancel{2}} = 2x^2 \end{array} \right|$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \ln x \cdot 2x^2 - \int \cancel{2x^2} \cdot \frac{1}{\cancel{x}} \cdot dx \rightarrow \cancel{\frac{2x^3}{3} \cdot \ln|x|}$$

$$= 2x^2 \ln x - 2\int x \cdot dx$$

$$= 2x^2 \ln x - \cancel{2}\frac{x^2}{\cancel{2}} + C$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

b) $$\int \ln x \, dx = \left| \begin{array}{lll} u = \ln x & \xrightarrow{\text{DER.}} & du = \frac{1}{x} \cdot dx \\ dv = dx & \xrightarrow{\text{INT.}} & v = x \end{array} \right| = x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot dx = x\ln x - x + C$$

$$(\square^n)' = n\square^{n-1} \cdot \square'$$

c) $$\int \ln^2 x \, dx = \left| \begin{array}{lll} u = \boxed{\ln^2 x} & \xrightarrow{\text{DER.}} & du = 2\ln x \cdot (\ln x)' \cdot dx = 2\ln x \cdot \frac{1}{x} dx \\ dv = dx & \xrightarrow{\text{INT}} & v = x \end{array} \right|$$

$$= x \cdot \ln^2 x - \int \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} \cdot dx$$

$$= x\ln^2 x - 2\underbrace{\int \ln x \cdot dx} = (*)$$

OPET PARCIJALNA:

$$\underbrace{\int \ln x \cdot dx} = \left| \begin{array}{lll} u = \ln x & \xrightarrow{\text{DER.}} & du = \frac{1}{x} dx \\ dv = dx & \xrightarrow{\text{INT.}} & v = x \end{array} \right| = x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot dx = \underbrace{x \cdot \ln x - x}$$

$$(*) = x\ln^2 x - 2 \cdot (x\ln x - x) + C$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

d) $$\int \frac{\ln x}{x^2} \, dx = \int \ln x \cdot x^{-2} \cdot dx = \left| \begin{array}{lll} u = \ln x & \xrightarrow{\text{DER.}} & du = \frac{1}{x} \cdot dx \\ dv = x^{-2} dx & \xrightarrow{\text{INT.}} & v = \frac{x^{-1}}{-1} = -\frac{1}{x} \end{array} \right|$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1}$$

$$= -\frac{1}{x}\ln x \overset{+}{-} \int -\frac{1}{x} \cdot \frac{1}{x} \cdot dx$$

$$= -\frac{1}{x}\ln x + \int x^{-2} \cdot dx$$

$$= -\frac{1}{x}\ln x \overbrace{-\frac{1}{x}} + C$$

3. TIP: KRUŽNI INTEGRAL

SVEJEDNO TKO POD (u) a tko POD (dv)

a) $\displaystyle\int e^x \sin x\, dx = \left| \begin{array}{ll} u = e^x \xrightarrow{\text{DER.}} & du = e^x \cdot dx \\ dv = \sin x \cdot dx \xrightarrow{\text{INT.}} & v = -\cos x \end{array} \right|$

$= -e^x \cos x \cancel{-} \int \cancel{-}\cos x \cdot e^x \cdot dx$

$= -e^x \cos x + \underline{\int e^x \cos x \cdot dx} = (*)$

OPET PARCIJALNA:

$\underline{\int e^x \cos x \cdot dx} = \left| \begin{array}{ll} u = e^x \xrightarrow{\text{DER.}} & du = e^x \cdot dx \\ dv = \cos x \cdot dx \xrightarrow{\text{INT.}} & v = \sin x \end{array} \right| = \underline{e^x \sin x - \int \sin x \cdot e^x \cdot dx}$

OSTATI DOSLJEDNI
Sa u i dv

→ ISTI IZRAZI OSTAJU POD NJIMA

IZJEDNAČIMO POČETAK I KRAJ

$\underbrace{\int e^x \cdot \sin x \cdot dx}_{I} = (*) = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \cdot dx}_{I}$

$I = ?$

$I = -e^x \cos x + e^x \sin x - I$

$I + I = -e^x \cdot \cos x + e^x \sin x$

$2I = -e^x \cos x + e^x \sin x \,/ : 2$

$\int e^x \sin x\, dx = I = \dfrac{-e^x \cos x + e^x \sin x}{2} + C$

$(\cos \square)' = -\sin \square \cdot \square'$

c) $\displaystyle\int \cos(\ln x)\, dx = \left| \begin{array}{ll} u = \cos(\boxed{\ln x}) \xrightarrow{\text{DER}} & du = -\sin(\ln x) \cdot (\ln x)' \cdot dx = -\sin(\ln x) \cdot \frac{1}{x} dx \\ dv = dx \xrightarrow{\text{INT.}} & v = x \end{array} \right|$

$= x \cdot \cos(\ln x) \cancel{-} \int \cancel{-} \cancel{x} \cdot \sin(\ln x) \cdot \frac{1}{\cancel{x}} \cdot dx$

$= x \cos(\ln x) + \underline{\int \sin(\ln x) \cdot dx} = (*)$

$\underline{\int \sin(\ln x) dx} = \left| \begin{array}{ll} u = \sin(\ln x) \xrightarrow{\text{DER.}} & du = \cos(\ln x) \cdot \frac{1}{x} \cdot dx \\ dv = dx \xrightarrow{\text{INT.}} & v = x \end{array} \right| = \underline{x \cdot \sin(\ln x) - \int \cancel{x} \cdot \cos(\ln x) \cdot \frac{1}{\cancel{x}} dx}$

$\underline{\int \cos(\ln x) \cdot dx} = (*) = x \cdot \cos(\ln x) + x \sin(\ln x) - \underline{\int \cos(\ln x) \cdot dx}$

$$\underbrace{\int \cos(\ln x)\cdot dx}_{I} = (*) = x\cdot\cos(\ln x) + x\sin(\ln x) - \underbrace{\int \cos(\ln x)\cdot dx}_{I}$$

$$I = x\cos(\ln x) + x\sin(\ln x) - I$$

$$2I = x\cos(\ln x) + x\sin(\ln x) \;/ :2$$

$$\int \cos(\ln x)dx = I = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$$

$$\int u\cdot dv = u\cdot v - \int v\cdot du \qquad (\ln x)' = \frac{1}{x}$$

d) $$\int \frac{1\cdot\ln x}{x}\,dx = \left| \begin{array}{lcl} u = \ln x & \xrightarrow{\text{DER.}} & 1\cdot du = \frac{1}{x}\cdot dx \\ dv = \frac{1}{x}\cdot dx & \xrightarrow{\text{INT.}} & v = \ln|x| = \ln x \end{array} \right|$$

$$\int \frac{1}{x}\cdot dx = \ln|x|$$

$|x| = x$ jer vidimo da je x pod integralom unutar ln što znači da x mora biti > 0.

$$= \ln x\cdot \ln x - \underbrace{\int \ln x\cdot\frac{1}{x}\cdot dx}_{\text{isto kao na početku}}$$

PA IZJEDNAČIMO POČETAK I KRAJ

$$\underbrace{\int \frac{\ln x}{x}\cdot dx}_{I} = \ln^2 x - \underbrace{\int \frac{\ln x}{x}\cdot dx}_{I}$$

$$I = \ln^2 x - I$$

$$2I = \ln^2 x \;|:2$$

$$\int \frac{\ln x}{x}dx = I = \frac{\ln^2 x}{2} + C$$

b) $\int \sin 2x \cos 3x\, dx =$ DZ

$$\int u\cdot dv = u\cdot v - \int v\cdot du \qquad (\sin\Box)' = \cos\Box\cdot\Box'$$

b) $$\int \sin 2x\cdot\cos 3x\cdot dx = \left| \begin{array}{lcl} \sin 2x = u & \xrightarrow{\text{der.}} & \cos 2x\cdot\underbrace{(2x)'}_{2}\cdot dx = 1\cdot du \\ & & 2\cos 2x\cdot dx = du \\ \cos 3x\cdot dx = dv & \xrightarrow{\text{int.}} & \end{array} \right.$$

$$\cos 3x \cdot dx = dv \xrightarrow{\text{int.}} \frac{1}{3}\cdot\sin 3x = v \qquad 2\cos 2x \cdot dx = du$$

$$u\cdot v - \int v\cdot du$$

$$= \sin 2x \cdot \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x \cdot 2\cos 2x \, dx$$

$$= \frac{1}{3}\cdot\sin 2x\cdot\sin 3x - \frac{2}{3}\cdot\underbrace{\int \sin 3x\cdot\cos 2x\cdot dx} = (*)$$

još jedna parcijalna

$$\int u\cdot dv = uv - \int v du$$

$$\int \sin 3x\cdot\cos 2x\cdot dx = \left| \begin{array}{ll} \cos 2x = u & \xrightarrow{\text{der.}} -\sin 2x\cdot(2x)'\cdot dx = du \\ & -2\sin 2x\cdot dx = du \\ \sin 3x\cdot dx = dv & \xrightarrow{\text{int.}} -\frac{1}{3}\cos 3x = v \end{array} \right|$$

$$= \cos 2x\cdot\left(-\frac{1}{3}\cos 3x\right) - \int\left(-\frac{1}{3}\cos 3x\cdot(-2\sin 2x)\right)dx$$

$$= \underline{-\frac{1}{3}\cos 2x\cdot\cos 3x - \frac{2}{3}\cdot\int\cos 3x\cdot\sin 2x\cdot dx}$$

$$=(*) = \frac{1}{3}\sin 2x\cdot\sin 3x - \frac{2}{3}\left(-\frac{1}{3}\cos 2x\cdot\cos 3x - \frac{2}{3}\cdot\int\cos 3x\cdot\sin 2x\cdot dx\right)$$

$$= \frac{1}{3}\sin 2x\cdot\sin 3x + \frac{2}{9}\cdot\cos 2x\cdot\cos 3x + \frac{4}{9}\int\cos 3x\cdot\sin 2x\cdot dx$$

IZJEDNAČIMO POČETAK I KRAJ I IZRAZIMO NEPOZNATI INTEGRAL I

$$\underbrace{\int \sin 2x\cdot\cos 3x\cdot dx}_{I} = \frac{1}{3}\sin 2x\cdot\sin 3x + \frac{2}{9}\cos 2x\cdot\cos 3x + \frac{4}{9}\cdot\underbrace{\int\cos 3x\cdot\sin 2x\, dx}_{I}$$

$$I = ?$$

$$I - \frac{4}{9}I = \frac{1}{3}\sin 2x\cdot\sin 3x + \frac{2}{9}\cos 2x\cdot\cos 3x \quad \Big| \cdot 9$$

$$9I - 4I = 3\sin 2x\cdot\sin 3x + 2\cos 2x\cdot\cos 3x$$

$$5I = 3\sin 2x\cdot\sin 3x + 2\cos 2x\cdot\cos 3x \quad \Big| :5$$

$$\int \sin 2x\cdot\cos 3x\, dx \quad I = \frac{3\sin 2x\cdot\sin 3x + 2\cos 2x\cdot\cos 3x}{5} + C$$